Relaxation of a Two-Specie Magnetofluid

L. C. Steinhauer
Redmond Plasma Physics Laboratory, University of Washington, Redmond, Washington 98052

A. Ishida
Department of Environmental Science, Faculty of Science, Niigata University, Ikarashi, Niigata 950-21 Japan
(Received 14 April 1997)

The relaxation theory of an ideal magnetofluid is developed for a multispecie magnetofluid. Its invariants are the self-helicities, one for each specie. Their "local" invariance in the ideal case follows from the helicity transport equation. The global forms of the self-helicities are investigated for a two-fluid (ion and electron), and their ruggedness in a weakly dissipative system is defended by cascade and selective decay arguments. In general the two-fluid theory predicts relaxed states with finite pressure and sheared flows. The familiar single-fluid relaxation theory, which admits only force-free states, is a reduced case of the present more general theory. [S0031-9007(97)04375-5]

PACS numbers: 52.30.Bt, 47.65.+a, 52.55.Dy

Knowledge of a system's invariants often leads to an elegant qualitative picture of its behavior, particularly that such constraints cause it to self-organize into *relaxed states* [1]. An example is ideal magnetohydrodynamics (MHD) where the invariance of the magnetic helicity has fostered successful predictions of self-organization by certain classes of magnetofluid into force-free states, i.e., equilibria with no coupling force between the system's fluid and field elements [2,3]. However, practical magnetofluids in space and in fusion experiments are not generally force free, i.e., they exhibit significant fluid pressure. Further, significant flows (not predicted by the MHD theory) are a nearly ubiquitous feature of practical plasmas. A more realistic formulation of a magnetofluid is a multifluid system, e.g., a *two-fluid* (ions and electrons). In a multifluid the invariants are the self-helicities (one for each specie), which are canonical composites of the fluid and magnetic momenta. MHD is simply the reduced case of a two-fluid in which the ion and electron responses are locked together; and the magnetic helicity is simply the electron self-helicity in the limit of massless electrons.

In this Letter we develop the two-fluid theory of relaxation. Helicity transport equations, which govern the "local" form of a helicity, are derived from the equations of motion and Maxwell's equations. This leads to the *ideal* invariance (dissipationless case) of the self-helicities. The global form of a self-helicity is the integral of its local form over the system volume. The ruggedness of the global self-helicities subject to a weak dissipation is defended by cascade and selective decay arguments. In this analysis, the fluid-field coupling in turbulent fluctuations plays an important role. Finally, relaxed states follow from minimizing the magnetofluid energy subject to constrained self-helicities. Inspection of the resulting Euler equations together with the steady equation of motion shows that non-negligible fluid pressure and sheared flows are common features of relaxed states. Throughout the discussion, we show the relationship of this more general theory to the familiar reduced theory based on MHD.

We begin by deriving equations for the evolution of two basic electromagnetic and two (for each specie) mechanical quantities. Consider the vector and scalar potentials (the existence of which is an implicit expression of Faraday's law): Use the definition $\partial \mathbf{A}/\partial t = -c\mathbf{E} + -c\nabla\phi$ ($\mathbf{A}$ and ϕ are the potentials); augment $\partial \mathbf{A}/\partial t$ to construct the total derivative with respect to specie α, $D_\alpha/Dt \equiv \partial/\partial t + \mathbf{u}_\alpha \cdot \nabla$; and replace the electric field in favor of the Lorentz force on specie α,

$$\mathbf{F}_\alpha = q_\alpha n_\alpha [\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}/c]. \tag{1}$$

Here $\alpha = i$ (ions), e (electrons) denotes the specie; q_α, n_α, $\mathbf{u}_\alpha$ are the charge, number density, and flow velocity; and $\mathbf{E}$, $\mathbf{B}$ are the electric and magnetic fields. Then

$$\frac{D_\alpha \mathbf{A}}{Dt} = \mathbf{u}_\alpha \times \mathbf{B} + \mathbf{u}_\alpha \cdot \nabla \mathbf{A} - \nabla(c\phi) - \frac{c}{q_\alpha}\left(\frac{\mathbf{F}_\alpha}{n_\alpha}\right). \tag{2}$$

Consider the explicit form of Faraday's law: Augment $\partial \mathbf{B}/\partial t$ to construct the total derivative; again replace $\mathbf{E}$ in favor of the Lorentz force, add $\mathbf{B}$ times the continuity equation $[n_\alpha D_\alpha(1/n_\alpha)Dt = \nabla \cdot \mathbf{u}_\alpha]$; and simplify using a vector identity for $\nabla \times (\mathbf{u}_\alpha \times \mathbf{B})$ and $\nabla \cdot \mathbf{B} = 0$. Then

$$n_\alpha \frac{D_\alpha}{Dt}\left(\frac{\mathbf{B}}{n_\alpha}\right) = \mathbf{B} \cdot \nabla \mathbf{u}_\alpha - \frac{c}{q_\alpha} \nabla \times \left(\frac{\mathbf{F}_\alpha}{n_\alpha}\right). \tag{3}$$

Consider the mechanical elements of the system. For barotropic species $[p_\alpha = p_\alpha(n_\alpha)]$, the equations of motion can be expressed as

$$m_\alpha \frac{D_\alpha \mathbf{u}_\alpha}{Dt} = -\nabla h_\alpha + \frac{\mathbf{F}_\alpha}{n_\alpha} + \frac{\mathbf{R}_\alpha}{n_\alpha}, \tag{4}$$

where $h_\alpha \equiv \int dp_\alpha/n_\alpha$, and $\mathbf{R}_\alpha$ is the frictional force density (resistive plus viscous). An equation for the fluid vorticity, $\boldsymbol{\omega}_\alpha \equiv \nabla \times \mathbf{u}_\alpha$, follows by taking the curl of

Eq. (4), adding ω_α times the continuity equation, and using vector identities for ∇u_α^2 and $\nabla \times (\mathbf{u}_\alpha \times \boldsymbol{\omega}_\alpha)$:

$$m_\alpha n_\alpha \frac{D_\alpha}{Dt}\left(\frac{\boldsymbol{\omega}_\alpha}{n_\alpha}\right) = m_\alpha \boldsymbol{\omega}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla \times \left(\frac{\mathbf{F}_\alpha}{n_\alpha}\right) + \nabla \times \left(\frac{\mathbf{R}_\alpha}{n_\alpha}\right). \quad (5)$$

We now have expressions for the evolution of two electromagnetic and two mechanical elements: $\mathbf{A}$ [Eq. (2)] and $\mathbf{u}_\alpha$ [Eq. (4)] are analogous, as are their curls, $\mathbf{B}$ [Eq. (3)] and $\boldsymbol{\omega}_\alpha$ [Eq. (5)]. In each equation is an electromechanical coupling term containing the specific (per particle) Lorentz force $\mathbf{F}_\alpha/n_\alpha$. (Here and elsewhere "specific" denotes per particle.)

The next step is to examine quadratic elements (specific helicities) in search of ideal invariants. From Eqs. (2) and (3), the transport equation for the magnetic helicity $\mathbf{A} \cdot \mathbf{B}/n_\alpha$ is

$$n_\alpha \frac{D_\alpha}{Dt}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{n_\alpha}\right) = \nabla \cdot \left[(\mathbf{u}_\alpha \cdot \mathbf{A} - c\phi)\mathbf{B} - \frac{c}{q_\alpha}\mathbf{A} \times \frac{\mathbf{F}_\alpha}{n_\alpha}\right] - \frac{2c}{q_\alpha}\mathbf{B} \cdot \frac{\mathbf{F}_\alpha}{n_\alpha}. \quad (6)$$

Similar transport equations can be found for the fluid helicity $\mathbf{u}_\alpha \cdot \boldsymbol{\omega}_\alpha/n_\alpha$, [from Eqs. (3),(4)] and the specific cross helicity $\mathbf{u}_\alpha \cdot \mathbf{B}/n_\alpha$ [from Eqs. (4),(5)]. Further quadratic elements are found by combining the electromagnetic and mechanical elements into the well-known canonical momenta $\mathbf{P}_\alpha = m_\alpha \mathbf{u}_\alpha + q_\alpha \mathbf{A}_\alpha/c$ and their curl, the canonical vorticity $\boldsymbol{\Omega}_\alpha = \nabla \times \mathbf{P}_\alpha$. Notably, in the evolution equations for $\mathbf{P}_\alpha$ and $\boldsymbol{\Omega}_\alpha$ [from Eqs. (2)–(5)] the electromagnetic coupling terms (Lorentz force) completely cancel: This hints that $\mathbf{P}_\alpha$ and $\boldsymbol{\Omega}_\alpha$ are the *natural* electromechanical quantities. The associated quadratic elements are the specific self-helicities $\mathbf{P}_\alpha \cdot \boldsymbol{\Omega}_\alpha/n_\alpha$. The self-helicity transport equations (one for each specie) follow [from Eqs. (2)–(5)]:

$$n_\alpha \frac{D_\alpha}{Dt}\left(\frac{\mathbf{P}_\alpha \cdot \boldsymbol{\Omega}_\alpha}{n_\alpha}\right) = \nabla \cdot \left[\left(\mathbf{u}_\alpha \cdot \mathbf{P}_\alpha - \frac{m_\alpha u_\alpha^2}{2} - q_\alpha \phi - h_\alpha\right)\boldsymbol{\Omega}_\alpha - \mathbf{P}_\alpha \times \frac{\mathbf{R}_\alpha}{n_\alpha}\right] + 2\boldsymbol{\Omega}_\alpha \cdot \frac{\mathbf{R}_\alpha}{n_\alpha}. \quad (7)$$

Related helicity transport equations have been derived elsewhere. In the *reduced* model of an ideal Ohm's law ($\alpha = i$ and $\mathbf{F}_i = 0$) Moffatt [4] derived transport equations for the magnetic [Eq. (6)] and cross helicities. For a pure fluid (no Lorentz force) Moffatt also derived the fluid helicity transport equation. Bhadra and Chu [5] derived the ion helicity transport equation [Eq. (7) with $\alpha = i$]. The forms in Eqs. (6) and (7) are new in that they display explicitly the electromechanical coupling, and they are referenced to a (any) specie.

Local invariance follows from the transport equations, which share a common form: The total derivative of a specific helicity is expressed as the sum of a divergence term, an electromagnetic coupling term, and a frictional term. Consider the ideal case, i.e., set the frictional term to zero; if the coupling term happens to vanish, then the helicity is invariant on the characteristic lines (or tubes) that are tangential to the vector in the divergence term. Then the helicity is *locally invariant* in the sense that it is constant on each characteristic line that doesn't intercept the system boundary. These characteristic lines convect with a particular specie. Consider the magnetic helicity [Eq. (6)] with $\alpha = i$: It is invariant only in the reduced case of *zero coupling force* $\mathbf{F}_i = 0$; this is identical to the ideal Ohm's law [see Eq. (1)]. Thus the magnetic helicity is locally invariant only in MHD (single-fluid). Its characteristic lines are the magnetic flux lines, and they convect with the ion specie. By contrast, the self-helicities [Eq. (7)] are invariant *without reducing assumptions:* This is because the electromagnetic coupling is absent from their transport equations. Their characteristic lines convect with the particular specie.

Global invariance concerns the global forms of a helicity and becomes important when weak dissipation is considered. The global form is the integral $\int n_\alpha \, d\tau$ ($d\tau$ is the volume element) of the specific helicity, i.e., summing over all particles (of a specie). Then, e.g., the specific magnetic helicity $\mathbf{A} \cdot \mathbf{B}/n_i$ ($\alpha = i$) becomes the *global* magnetic helicity $K_m = \int \mathbf{A} \cdot \mathbf{B} \, d\tau$. We find the evolution of a global helicity from its transport equation [e.g., for K_m, Eq. (6) with $\alpha = i, \mathbf{F}_i = 0$] as follows: Take the volume integral $\int d\tau$, apply continuity $D_\alpha(n_\alpha d\tau)/Dt = 0$; and convert the integral of the divergence to a surface integral. For appropriate boundary conditions, the surface integral vanishes (e.g., for K_m, no normal component of $\mathbf{B}$ at the boundary). This leads to, e.g., $dK_m/dt = 0$ so that $K_m =$ const. Global self-helicity invariance follows by applying the same procedure to Eq. (7).

We have established the ideal invariance of the self-helicities. However, a useful relaxation theory rests on two further requirements: (1) The "direction" of the relaxation must be toward larger size structures; and (2) the invariants must be *rugged,* i.e., the K_α must decay more slowly than the magnetofluid energy W_{mf} in the presence of a weak dissipation. We address the former by a cascade argument based on the spectral form of the Fourier-analyzed quantities $\tilde{W}_{\mathrm{mf}}, \tilde{K}_\alpha$. We address the latter by selective decay arguments based on the spectral forms and on the decay rates, $dW_{\mathrm{mf}}/dt, dK_\alpha/dt,$ in thin reconnection layers. Since the self-helicities unite the mechanical and electromagnetic elements, the "fluid-field" coupling is needed. For this we employ linear wave theory. This is equivalent to adopting the well-known paradigm of quasilinear theory, i.e., that the frequencies of linear theory apply even though nonlinear processes ensue.

Consider the fluid-field coupling in the case of massless electrons: This excludes high-frequency phenomena (plasma frequency, electron cyclotron frequency) and assures quasineutrality $n_i = n_e = n$. The "organized" energy form is the magnetofluid energy $W_{\rm mf}$, which sums the flow kinetic and magnetic energies [4,6]; note that the electrostatic energy is negligible when high-frequency phenomena are excluded. The magnetofluid energy and the self-helicities with their Fourier-analyzed forms are

$$W_{\rm mf} = \int d\tau \left(\frac{1}{2}\, m_i n u_i^2 + \frac{B^2}{8\pi} \right) \rightarrow \tilde{W}_{\rm mf} = \frac{1}{8\pi} \left(\frac{1}{\ell_c^2} \left| \frac{m_i c \tilde{\mathbf{u}}_i}{e} \right|^2 + k^2 |\tilde{A}|^2 \right), \tag{8}$$

$$K_\alpha = \frac{q_\alpha^2}{8\pi c^2} \int d\tau \mathbf{P}_\alpha \cdot (\nabla \times \mathbf{P}_\alpha) \rightarrow \tilde{K}_\alpha \leq \frac{k}{8\pi} \left| \frac{m_\alpha c}{q_\alpha}\, \tilde{\mathbf{u}}_\alpha + \tilde{A} \right|^2 . \tag{9}$$

Fourier-analyzed quantities $[\exp(i\boldsymbol{k} \cdot \boldsymbol{r} - i\omega t)]$ are denoted by a tilde; $\mathbf{k}$ is the wave vector; the "$\leq$" reflects the Schwarz inequality. We have also introduced the factor $q_\alpha^2/8\pi c^2$ in the self-helicities (giving them energy-length units) and the collisionless skin depth $\ell_c = c/\omega_{pi} = (m_i c^2/4\pi e^2 n_0)^{1/2}$ where n_0 is the ambient density. Although K_e and K_m differ only by a constant factor for massless electrons, K_e rather than K_m is the proper "electron" invariant in a two-fluid. In the case of cold, nondrifting species the ion and electron responses to the field $\tilde{A}$ are

$$\frac{m_i c}{e}\, \tilde{\mathbf{u}}_i = \frac{\omega}{\omega_{\rm ci} - \omega^2} \left(\omega \tilde{A} - i\omega_{\rm ci} \frac{\mathbf{B}_0}{B_0} \times \tilde{\mathbf{A}} \right), \tag{10}$$

$$\frac{m_i c}{e}\, \tilde{\mathbf{u}}_e = -i \frac{\omega}{\omega_{\rm ci}} \frac{\mathbf{B}_0}{B_0} \times \tilde{\mathbf{A}}, \tag{11}$$

where $\omega_{\rm ci} = eB_0/m_i c$ is the ion cyclotron frequency and $\mathbf{B}_0$ is the ambient field. Inspection of Eqs. (10) and (11) shows that the ion and electron responses lock together for low frequencies $\omega \ll \omega_{\rm ci}$, i.e., the MHD model is valid. However, for higher frequencies $\omega \geq O(\omega_{\rm ci})$ the ion and electron responses differ markedly, i.e., the two-fluid model is necessary. An investigation of the dispersion relation (see, e.g., [7]) uncovers three propagating waves: the R wave ($k_\parallel$, *right* circularly polarized); the L wave ($k_\parallel$, *left* circularly polarized); and the magnetosonic wave ($k_\perp$, $\tilde{\mathbf{E}} \perp \mathbf{B}_0$). Here $\parallel$ and $\perp$ denote parallel and perpendicular to $\mathbf{B}_0$. For low k ($k\ell_c < 1$) each has Alfven-like frequency $\omega \approx k v_A = \omega_{\rm ci} k \ell_c$ (v_A is the Alfven speed) and lie in the low-frequency MHD regime. For high k ($k\ell_c > 1$) all responses require the two-fluid treatment: $\omega \approx \omega_{\rm ci}(k\ell_c)^2$, R wave; $\omega \approx \omega_{\rm ci}(k\ell_c)$, magnetosonic wave; and $\omega \approx \omega_{\rm ci}$, L wave.

Consider now the *cascade argument.* Since a two-fluid has a multiplicity of waves, two kinds of selection occur: intermode selection and spectral selection (intramode). In the former the selection favors the lowest energy wave. For low k the energy is about the same for all three waves, $\tilde{W}_{\rm mf} \approx k^2|\tilde{A}|^2/4\pi$, while at high k the R wave ("whistler") for which $\tilde{W}_{\rm mf} \approx k^2|\tilde{A}|^2/8\pi$ is energetically favorable. For low k there is equipartition between magnetic and the flow kinetic energies. This, with our observation that MHD behavior applies for low k, is consistent with the equipartition well known in MHD simulations. For high k, however, the flow kinetic energy is much less than the magnetic energy. The self-helicities are (low k), $\tilde{K}_e \approx \tilde{K}_i = k|\tilde{A}|^2/8\pi$, and (high k, R wave), $\tilde{K}_e = k|\tilde{A}|^2/8\pi$ and $\tilde{K}_i \approx (k\ell_c)^{-4} k|\tilde{A}|^2/8\pi$. Then for all ranges of k and the energetically favored wave the following inequalities hold:

$$\tilde{W}_{\rm mf}(k) \geq k\tilde{K}_\alpha(k); \qquad \alpha = i, e. \tag{12}$$

With these relationships in hand we can consider the spectral selection, which leads to the *cascade argument.* Here we apply the argument of Frisch *et al.* [6]. Having established the inequality $\tilde{W}(k) \geq k\tilde{K}(k)$, they showed that an *ideal* transition (one in which the global energy and helicity are preserved) can only lead toward *lower* k, i.e., toward larger-scale structures. This is an *inverse* cascade. Frisch applied this argument to $\tilde{W}_m, \tilde{K}_m$ (MHD model), but it applies equally to the magnetofluid energy and the self-helicities in view of the inequalities [Eq. (12)]. This argument does not apply to the cross and fluid helicities since they do not satisfy inequalities of the form in Eq. (12).

Observe two differences between the MHD and two-fluid models. (1) If one adopts the MHD model (ideal Ohm's law), then for low k, $\tilde{W}_{\rm mf} \propto k^2$ and $\tilde{K}_i \propto k$, while for high k (again) $\tilde{W}_{\rm mf} \propto k^2$, but $\tilde{K}_i \propto k^3$; therefore Eq. (12), on which the inverse cascade argument rests, doesn't hold for high k. It is unlikely then that a MHD simulation will predict invariant K_i, and particularly so since relaxation proceeds at thin reconnection layers corresponding to high k. Therefore the appearance of rugged ion helicity requires a two-fluid treatment. (2) Both $\tilde{K}_e$ and $\tilde{K}_m$ are rugged invariants in their particular contexts (two-fluid, MHD, respectively); however, K_e is on a firmer footing since it is based on the more general plasma model, and it doesn't rest on an artificial assumption (ideal Ohm's law).

The other argument supporting the relaxation model *selective decay,* which was articulated by Montgomery [8] and others (e.g., [2]), addresses the issue of ruggedness: in the presence of weak dissipation, are the self-helicities *more invariant* than the magnetofluid energy. The simplest selective decay argument springs from Eq. (12). Since $\tilde{W}_{\rm mf}$ is proportional to a higher power of k than $\tilde{K}_\alpha$, its spectrum should peak at a higher k. Then since dissipation is stronger at higher k (smaller scale) the magnetofluid energy should be dissipated faster than either of the self-helicities. Indeed for ions, this tendency is accentuated in view of the extreme smallness of $\tilde{K}_i$ at high k.

The second formulation of the selective decay argument compares the decay rates:

$$\frac{dW_{\mathrm{mf}}}{dt} = \int d\tau\Big(\eta j^2 + \sum_\alpha \nu_\alpha |\nabla u_\alpha|^2\Big); \qquad (13)$$

$$\frac{dK_\alpha}{dt} = -\frac{c^2}{4\pi q_\alpha^2}\int d\tau \mathbf{\Omega}_\alpha \cdot \Big(-q_\alpha \eta \mathbf{j} + \frac{\nu_\alpha}{n}\nabla^2 u_\alpha\Big); \qquad (14)$$

where $\mathbf{j}, \nu_\alpha \nabla^2 \mathbf{u}_\alpha$ are the current density and viscous stress, respectively. Define $1/k_R$ as the nominal reconnection layer thickness ($k_R \ell_c \gg 1$ for thin layers). Then

$$dW_{\mathrm{mf}}/dt \sim k_R^2 \frac{\eta c^2}{4\pi}\frac{|\tilde{B}|^2}{8\pi}\Big[1 + \frac{\mu}{k_R^2 \ell_c^2}\Big], \qquad (15)$$

$$dK_e/dt \sim k_R \frac{\eta c^2}{4\pi}\frac{|\tilde{B}|^2}{8\pi}, \qquad (16)$$

$$dK_i/dt \sim k_R \frac{\eta c^2}{4\pi}\frac{|\tilde{B}|^2}{8\pi}\Big[2 + \frac{\mu}{k_R^2 \ell_c^2}\Big]. \qquad (17)$$

Here $\tilde{B}$ is the perturbed magnetic field; the electron viscosity has been neglected; and the terms with $\mu \approx (m_i/2m_e)^{1/2} \approx 40$ represent the ion viscosity. The predominant scalings at high k_R imply that the energy decay rate $dW_{\mathrm{mf}}/dt \propto k_R^2$ is stronger than that of the self-helicities $dK_\alpha/dt \propto k_R$. The same proportionalities appear in the decay of W_m and K_m [2].

Having established the ruggedness of self-helicities, we are free to find relaxed states by solving the constrained variational problem $\delta W_{\mathrm{mf}} - \lambda_i \delta K_i - \lambda_e \delta K_e = 0$, where λ_i, λ_e are Lagrange multipliers. The resulting Euler equation for each specie are

$$\lambda_\alpha \mathbf{\Omega}_\alpha = (2\pi q_\alpha/c^2)\mathbf{j}_\alpha . \qquad (18)$$

Observe the following properties: (1) For massless electrons ($\alpha = e, m_e = 0$) the left side of Eq. (18) is proportional to the magnetic field, so that $\mathbf{j}_e \propto \lambda_e \mathbf{B}$; thus in the absence of ion currents ($\mathbf{j} = \mathbf{j}_e$), the equilibrium is *force free,* as in the MHD theory. In general, of course, the ion current is nonzero. (2) For ions ($\alpha = i$) the $\nabla \times \mathbf{u}_i$ term on the left side of Eq. (18) (recall that $\mathbf{\Omega}_i = m_i \nabla \times \mathbf{u}_i + e\mathbf{B}/c$) implies that significant sheared ion flows can arise in relaxed states. The pressure is found by summing the species equations of motion. For relaxed equilibria [Eq. (18)] this leads to a Bernoulli equation

$$\nabla p = -m_i n \nabla u_i^2/2, \qquad (19)$$

where p is the sum of the electron and ion pressures. An important consequence of Eq. (19) is that finite-pressure relaxed states can exist, but only with velocity shear (or more generally with $\nabla u_i^2 \neq 0$). It was noted earlier that the ion equation [Eq. (18), $\alpha = i$] allows significantly sheared flows.

The relaxation theory of a two-specie magnetofluid synthesizes several familiar concepts: relaxation; multi-fluid magnetohydrodynamics; and the canonical momenta. Its invariants are the self-helicities, based on the canonical momenta. These naturally conjoin the mechanical and electromagnetic elements in that their invariance requires no artificial assumptions about the electromechanical coupling. The range of two-fluid relaxed states includes both static force-free equilibria and equilibria with significant pressure and sheared flow. The familiar MHD (single-fluid) relaxation theory is a reduced case of the present more general two-fluid treatment. Since the MHD theory assumes no electromechanical coupling force on the ions (ideal Ohm's law), it is not surprising that it predicts only force-free states. A forthcoming work will show that two-fluid relaxed states include equilibria resembling field-reversed configurations, reversed-shear tokamaks, reversed-field pinches, and spheromaks, all of which are magnetic plasma configurations relevant to fusion. Future directions for the two-fluid theory should include the following: (1) Comparison of relaxed states with space and laboratory magnetofluids; (2) a two-fluid simulation to verify the preservation of both self-helicities (the two-fluid model is essential for the invariance of K_i); and (3) addition of finite ion orbit effects (not included in the two-fluid model), possibly using the gyroviscosity.

The authors wish to thank Ravi Sudan for suggesting the self-helicity invariants.

[1] See review by M.R. Brown, J. Plasma Phys. **52**, 203 (1997).
[2] J.B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974).
[3] J.B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).
[4] H.K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[5] D.K. Bhadra and C. Chu, J. Plasma Phys. **33**, 257 (1985).
[6] U. Frisch, A. Pouquet, J. Leorat, and A. Mazure, J. Fluid Mech. **68**, 769 (1975).
[7] N.A. Krall and A.W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973), Chap. IV.
[8] D. Montgomery, L. Turner, and G. Vahala, Phys. Fluids **21**, 757 (1978).