Noiseless Quantum Codes

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In this paper we study a model quantum register $\mathcal{R}$ made of N replicas (cells) of a given finite-dimensional quantum system S. Assuming that all cells are coupled with a common environment with equal strength we show that, for N large enough, in the Hilbert space of $\mathcal{R}$ there exists a linear subspace C_N which is dynamically decoupled from the environment. The states in C_N evolve unitarily and are therefore decoherence-dissipation free. The space C_N realizes a noiseless quantum code in which information can be stored, in principle, for an arbitrarily long time without being affected by errors. [S0031-9007(97)04311-1]

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Since the early days of quantum computation [1] theory it has been clear that maintaining quantum coherence in any computing system is an essential requirement in order to fully exploit the new possibilities opened by quantum mechanics. This issue is known as the *decoherence* problem [2]. Indeed, any real-life device unavoidably interacts with its environment, which is, typically, made by a huge amount of uncontrollable degrees of freedom. This interaction causes a corruption of the information stored in the system as well as errors in computation steps that eventually lead to wrong outputs. One of the possible approaches for overcoming such difficulty, in analogy with classical computation, is to resort to redundancy in encoding information, by means of the so-called *error correcting codes* (ECC). In these schemes—pioneered in [3] and raised to a high level of mathematical sophistication in [4]—information is encoded in linear subspaces C (codes) of the total Hilbert space in such a way that "errors" induced by the interaction with the environment can be detected and corrected. The essential point is that the detection of errors, if they belong to the class of errors correctable by the given code, should be performed without gaining any information about the actual state of the computing system prior to corruption. Otherwise, this would result in a further decoherence. The ECC approach can thus be considered as a sort of *active* stabilization of a quantum state in which, by monitoring the system and conditionally carrying on suitable operations, one prevents the loss of information. The typical system considered in quantum-information context is a *N-qubit register* $\mathcal{R}$ made of N replicas of a two-level system S (the qubit). In the ECC literature, once more in analogy with the classical case, it is assumed that each qubit of $\mathcal{R}$ is coupled with an independent environment.

In this Letter we will show that the so far neglected case in which all the qubits can be considered symmetrically coupled with the same environment might provide a new strategy in the struggle for preserving quantum coherence. The idea is that, in the presence of such a "coherent" environmental noise, one can design states that are hardly corrupted rather than states that can be easily corrected. In other words, the present approach consists in a *passive* (i.e., intrinsic) stabilization of quantum information, and in this sense it is complementary to EC. The resulting codes could be called *error avoiding*. Furthermore, from the broader point of view of the theory of open quantum systems, our result shows a systematic way of building nontrivial models in which dynamical symmetry allows unitary evolution of a subspace while the remaining part of the Hilbert space gets strongly entangled with the environment. In the following we first briefly recall the basic mechanism of decoherence. If $\mathcal{H}_S, \mathcal{H}_B$ denote, respectively, the system and the environment Hilbert spaces, the total Hilbert space is given by the tensor product $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. Let ρ_S (ρ_B) be a state over $\mathcal{H}_S$ ($\mathcal{H}_B$) (i.e., $\rho_\alpha \in \mathrm{End}(\mathcal{H}_\alpha), \rho_\alpha = \rho_\alpha^\dagger, \rho_\alpha \geq 0, \mathrm{tr}(\rho_\alpha) = 1, \alpha = S, B$). According to quantum mechanics, time evolution of the overall (closed) system is unitary; therefore, if $\rho(0) = \rho_S \otimes \rho_B$ is the initial state, then for any $t \geq 0$ one has $\rho(t) = U_t \rho(0) U_t^\dagger$, ($U_t^{-1} = U_t^\dagger$). The induced (Liouvillian) evolution on $\mathcal{H}_S$ (open) is given by $L_t^{\rho_B} : \rho_S \rightarrow \mathrm{tr}^B \rho(t)$, where tr^B denotes partial trace over $\mathcal{H}_B$. The crucial point is that, even if ρ_S is a pure state ($\rho_S^2 = \rho_S$), in a very short time it gets entangled with the bath and becomes mixed ($\rho_S^2 \neq \rho_S$). Typically, in a suitable $\mathcal{H}_S$ basis, the off-diagonal elements of ρ_S behave like $\exp(-t/\tau_{\mathrm{Deco}})$. The energy $\hbar \tau_{\mathrm{Deco}}^{-1}$ is a measure of the rate at which the information loss occurs. If an EC strategy is not used, τ_{Deco} sets an upper bound to the duration of any reliable computation. Notice that this mechanism, due to quantum fluctuations, is active at finite as well as zero temperature and does not necessarily imply that dissipation takes place. Let us then begin by considering a simple example, important for quantum information applications—N identical two-level systems (N-qubit register) coupled with a *single*

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thermal bath described by a collection of noninteracting linear oscillators. The Hamiltonian of the register (bath) is given by $H_S = \epsilon \sum_{i=1}^N \sigma_i^z$, ($H_B = \sum_k \omega_k b_k^\dagger b_k$). The bath-register interaction Hamiltonian is

$$H_I = \sum_{k,i=1}^{N} (g_{ki}\sigma_i^+ b_k + f_{ki}\sigma_i^+ b^\dagger + h_{ki}\sigma_i^z b_k + \text{H.c.}) . \quad (1)$$

The operators $\{\sigma_i^\alpha\}$ span N local sl(2) algebras

$$[\sigma_i^z, \sigma_j^\pm] = \pm \delta_{ij}\sigma_i^\pm, \qquad [\sigma_i^+, \sigma_j^-] = 2\delta_{ij}\sigma_i^z . \quad (2)$$

These commutation relations make clear the physical meaning of the interaction (1) in terms of elementary processes: The first (second) term describes the excitation of the qubit by the absorption (emission) of a bath mode with probability amplitude f_{ki} (g_{ki}). This (together with the conjugate processes) is the dissipative part of the interaction, responsible for the (irreversible) exchange of energy between register and bath. The third term in Eq. (1) is a conservative coupling that induces pure dephasing between states corresponding to different eigenvalues of operators $\{\sigma_i^z\}$. Now we make the basic physical assumption: The coupling functions g_{kj}, f_{kj}, h_{kj} *do not depend on the replica index* j. This is a generalization of the Dicke limit of quantum optics [5]. Such an assumption can be justified if the replicas have very close spatial positions with respect to the bath coherence length ξ_C. Indeed if, for instance, $g_{kj} = g_k e^{ikR_j}$ ($\{R_j\}$ denoting the replica positions), with g_k not negligible for $k \le \xi_C^{-1}$, one has to impose $e^{ika} \simeq 1$, a being the typical distance between the replicas. In other terms, in (1) the systems have to be coupled only with bath modes with $k \ll a^{-1}$. Now the whole Hamiltonian $H_{\rm SB} = H_S + H_B + H_I$, can be written by means of the global operators $S^\alpha = \sum_{i=1}^N \sigma_i^\alpha (\alpha = \pm, z)$. In particular, the interaction reads

$$H_I = \sum_k (g_k S^+ b_k + f_k S^- b_k^\dagger + h_k S^z b_k + \text{H.c.}) . \quad (3)$$

In such a case only the global generators S^α are effectively coupled with the environment, whereby only collective coherent modes of $\mathcal{R}$ are involved in the system dynamics. Despite this simplification, the model described by $H_{\rm SB}$ is, in general, a nonintegrable interacting system, and therefore nontrivial. The exact eigenstates of $H_{\rm SB}$ are generally given by highly entangled states of $\mathcal{R}$ and the bath. Nevertheless, since the S^α's span an algebra isomorphic with sl(2), for N even, *one can build a family of eigenstates of $H_{\rm SB}$ given by simple tensor products*. For $N = 2$ let us consider the singlet state $|\psi\rangle = 2^{-1/2}(|01\rangle - |10\rangle)$ (in an obvious binary notation): since $S^\alpha|\psi\rangle = 0, (\alpha = \pm, z)$, one has that for *every* $|\psi_B\rangle \in \mathcal{H}_B$ the state $|\psi\rangle \otimes |\psi_B\rangle$ is annihilated by the interaction Hamiltonian. Moreover, it is a $H_{\rm SB}$ eigenstate if $|\psi_B\rangle$ is a H_B eigenstate, namely, $|\psi_B\rangle$ has the form $|\psi_B\rangle = \prod_j b_{k_j}^\dagger |0\rangle_B \equiv |K\rangle$, where $K = (k_1, \ldots, k_n)$ denotes a n tuple of wave vectors k ($n \in \mathbf{N}$). Let E_K be the corresponding eigenvalue. For $N > 2$ (even) the existence of states $|\psi_j^{(N)}\rangle$ behaving like the singlet $|\psi\rangle$ is ensured by the elementary sl(2) representation theory. The irreducible representations (irreps) $\mathcal{D}_j$ of sl(2) are labeled by the total angular momentum eigenvalue j and are $(2j+1)$ dimensional. When $j = 0$, one has one-dimensional representations. The corresponding states (singlets) are the many-qubit generalization of $|\psi\rangle$. In general, given a (reducible) representation $\mathcal{D}$ of sl(2), one has the Clebsch-Gordan (CG) decomposition in terms of the $\mathcal{D}_j$'s

$$\mathcal{D}^{\otimes N} = \bigoplus_{j\in J} n_j \mathcal{D}_j , \quad (4)$$

the integer n_j being the multiplicity with which $\mathcal{D}_j$ occurs in the resolution of $\mathcal{D}$. The S^α's realize a (reducible) representation $\mathcal{D}_{1/2}^{\otimes N}$ of sl(2) in $\mathcal{H}_S \cong (\mathbf{C}^2)^{\otimes N}$, that is the N-fold tensor product of the (defining) two-dimensional representation $\mathcal{D}_{1/2}$. The Clebsch-Gordan series reads for $N = 2$, 4, and 6

$$\mathcal{D}_{1/2}^{\otimes 2} = \mathcal{D}_1 \oplus \mathcal{D}_0, \qquad \mathcal{D}_{1/2}^{\otimes 4} = \mathcal{D}_2 \oplus 3\mathcal{D}_1 \oplus 2\mathcal{D}_0,$$
$$\mathcal{D}_{1/2}^{\otimes 6} = \mathcal{D}_3 \oplus 5\mathcal{D}_2 \oplus 9D_1 \oplus 5\mathcal{D}_0 .$$

Therefore, if $n(N)$ denotes the multiplicity of the $j = 0$ representation, one has $n(2) = 1, n(4) = 2, n(6) = 5$. Let C_N be the $n(N)$-dimensional space spanned by the singlets: It is immediate—by reasoning as in the $N = 2$ case—to verify that, if $|\psi^{(N)}\rangle \in C_N$ then $\forall |\psi_B\rangle \in \mathcal{H}_B$, one has $H_I|\psi^{(N)}\rangle \otimes |\psi_B\rangle = 0$. From this property comes the following result:

Theorem 1.—Let $\mathcal{M}_N$ be the manifold of states built over the singlet space C_N. If $\rho = \sum_{ij} R_{ij}|\psi_i^{(N)}\rangle\langle\psi_j^{(N)}| \in \mathcal{M}_N$, then for any initial bath state ρ_B one has $L_t^{\rho_B}(\rho) = \rho, \forall t > 0$.

Proof.—Let $\rho_B = \sum_{K',K} R_{K'K}|K'\rangle\langle K|$, and $\rho = \sum_{ij} \rho_{ij}|\psi_i^{(N)}\rangle\langle\psi_j^{(N)}|$. Then, if $\rho(t) = U(t)\rho \otimes \rho_B U^\dagger(t)$,

$$\begin{aligned}\rho(t) &= \sum_{ij,K'K} \rho_{ij} R_{K'K} \\ &\quad \times U(t)|\psi_i^{(N)}\rangle \otimes |K'\rangle(\langle\psi_j^{(N)}| \otimes \langle K|)U^\dagger(t) \\ &= \sum_{ij,K'K} \rho_{ij} R_{K'K} |\psi_i^{(N)}\rangle \otimes |K'\rangle \\ &\quad \times \langle\psi_j^{(N)}| \otimes \langle K| e^{-i(E_{K'}-E_K)t},\end{aligned} \quad (5)$$

and taking the trace over the bath one gets

$$\begin{aligned}\rho_t &= \sum_{ij} \rho_{ij}|\psi_i^{(N)}\rangle\langle\psi_j^{(N)}| \\ &\quad \times \sum_{K'K} R_{K'K} e^{-i(E_{K'}-E_K)t} \text{tr}^B |K'\rangle\langle K| \\ &= \sum_{ij} \rho_{ij}|\psi_i^{(N)}\rangle\langle\psi_j^{(N)}| \sum_K R_{KK} = \rho ,\end{aligned} \quad (6)$$

where we used $\mathrm{tr}(|K'\rangle\langle K|) = \delta_{K'K}B$, and $\sum_K R_{KK} = \mathrm{tr}^B \rho_B = 1$.

The result stated by theorem 1 can be rephrased in the following way which emphasizes its strength: In the manifold of the states over $\mathcal{H}_S$ there exists a submanifold $\mathcal{M}_N$ of fixed points (stationary states) of the Liouvillian evolution. The dynamics over $\mathcal{M}_N$ is therefore *a fortiori* unitary. Notice that this result relies only on algebra-theoretic properties and not on any "perturbative" assumptions; in other words, it holds for arbitrary strength of the system-bath coupling. This suggests the possibility of encoding in $\mathcal{M}_N$ decoherence-free information, namely, the states of $\mathcal{M}_N$ realize a *noiseless quantum code*. For example, a (nonorthogonal) basis of C_4 is

$$|\psi_1^{(4)}\rangle = 2^{-1}(|1001\rangle - |0101\rangle + |0110\rangle - |1010\rangle),$$
$$|\psi_2^{(4)}\rangle = 2^{-1}(|1001\rangle - |0011\rangle + |0110\rangle - |1100\rangle).$$

Orthonormalizing $|\psi_j^{(4)}\rangle$ ($j = 1, 2$), one generates a *noiseless* qubit.

It is remarkable that this result can be considerably generalized in many respects. In the sequel we shall discuss such generalizations with no proofs; the mathematical details will be given elsewhere [6]. Basic ingredients are the concept of *dynamical algebra* [7] and the standard Lie-algebra representation theory tools [8]. In what follows by dynamical algebra $\mathcal{A}_S$ of a system, with Hamiltonian $H \in \mathrm{End}(\mathcal{H})$, we mean the minimal Lie subalgebra of $\mathrm{gl}(\mathcal{H})$, such that (i) $H \in \mathcal{A}_S$ and (ii) H can be cast in diagonal form (i.e., linear combination of the Cartan generators) by means of a Lie-algebra inner automorphism $\Phi : \mathcal{A}_S \to \mathcal{A}_S$ (generalized Bogolubov rotation).

A system S endowed with the dynamical algebra $\mathcal{A}_S$ with Chevalley basis, $\{e_\alpha, e_{-\alpha}, h_\alpha\}_{\alpha=1}^r$, can be thought of as a collection of elementary excitations generated over the "vacuum" by the raising operators e_α of $\mathcal{A}_S$. These excitations are destroyed by the lowering generators $e_{-\alpha} = e_\alpha^\dagger$. The Cartan (Abelian) subalgebra spanned by the h_α's acts diagonally. The sl(2) (qubit) case corresponds to $r = 1$, the e_α's ($e_\alpha^\dagger$'s) are the analog of σ^- (σ^+), whereas the h_α's correspond to σ^z. The Hamiltonian can be written, in view of (ii) above, in a diagonal form as $H = \sum_{\alpha=1}^r \epsilon_\alpha h_\alpha$. We consider now the N noninteracting replicas of S. The Hilbert space becomes $\mathcal{H}_S = \mathcal{H}^{\otimes N}$, with $\dim(\mathcal{H}_S) = d^N$. As in the qubit case it is useful to introduce the global operators $X_\alpha \equiv \sum_{j=1}^N x_\alpha^j$, where x_α^i acts as $x_\alpha \in \mathcal{A}_S$ in the ith factor of the tensor product, and as the identity in the remaining factors. The operators $\{E_\alpha, E_{-\alpha}, H_\alpha\}$ span an algebra isomorphic with $\mathcal{A}_S$. The global Hamiltonian of the register can be written then in terms of the generators H_α of the Cartan subalgebra of $\mathcal{A}_S$ as $H_S = \sum_{\alpha=1}^r \epsilon_\alpha H_\alpha$. We assume that the system-bath interaction couples directly the bosonic modes with the elementary excitations of the system. The interaction Hamiltonian has the form, analog to that of Eq. (3),

$$H_I = \sum_{k\alpha} \tau_\alpha (g_k^\alpha E_\alpha b_k + f_k^\alpha E_\alpha^\dagger b_k^\dagger + h_k^\alpha H_\alpha b_k + \text{H.c. }),$$

where we have already assumed the replica symmetry of the coupling functions. The elementary processes associated with this H_I have the same interpretation as in the qubit case. As far as our basic result is concerned the assumption—physically motivated—that S is bilinearly coupled with the bath by the Chevalley basis operators of the $\mathcal{A}_S^i$'s is not restrictive. Indeed, if one were given as initial data not the dynamical algebra $\mathcal{A}_S$ but the system operators coupled with the environment, as well as H, one could reconstruct $\mathcal{A}_S$ by closing all possible commutation relations. In the generic case the algebra $\mathcal{A}_S$ thus generated turns out to be semisimple and acts irreducibly on $\mathcal{H}$. Since the global operators span an algebra *isomorphic* with $\mathcal{A}_S$, one can use the $\mathcal{A}_S$ representation theory to split $\mathcal{H}_{\mathrm{SB}} = \mathcal{H}_S \otimes \mathcal{H}_B$ according to the irreps of $\mathcal{A}_S$. In the following, without loss of generality, we let $\mathcal{A}_S \equiv \mathrm{sl}(r + 1)$ and let $\mathcal{D}$ denote the defining representation of $\mathcal{A}_S$ in $\mathcal{H}$ ($d = \dim\mathcal{H} = r + 1$). We need to consider the CG series of the N-fold tensor product representation of $\mathcal{A}_S$ in $\mathcal{H}^{\otimes N}$. It has the same form of (4), the set $\mathcal{J}$ being now the label set for the irreps of $\mathrm{sl}(r + 1)$, and n_j the multiplicity of the irrep $\mathcal{D}_j$.

An easy way to compute the CG series is to resort to the Young diagrams which relate the representation theory of $\mathrm{sl}(r + 1)$ with that of the symmetric group S_N [8]. Each Young diagram $\mathcal{Y}$ is associated with an irrep of S_N. Indeed, if $|\psi\rangle = \otimes_{j=1}^N |\psi_j\rangle$ is a basis vector of $\mathcal{H}^{\otimes N}$, the formula $\sigma|\psi\rangle = \otimes_{j=1}^N |\psi_{\sigma(j)}\rangle$ defines, for any $\sigma \in S_N$, by linear extension, a natural S_N action over $\mathcal{H}^{\otimes N}$. The multiplicities n_j are the dimensions of the S_N irreps associated with $\mathcal{Y}$. The dimension d_j of $\mathcal{D}_j$ is given by the number of different Young tableaux that one can obtain from $\mathcal{Y}$, and is equal to the multiplicity of the associated S_N irrep. For $N = r + 1$ one finds, with multiplicity one, the (fundamental) antisymmetric representation $\mathcal{D}_A$, associated with the $(r + 1, 1)$ Young diagram with just one column of N boxes [we use the notation (n, m) for the *rectangular* Young diagram with n rows and m columns]. $\mathcal{D}_A$ is one dimensional and given by the vector,

$$|\psi_A\rangle = N!^{-1/2} \sum_{\sigma \in S_N} (-1)^{|\sigma|} \sigma \otimes_{j=1}^N |j\rangle,$$

$\{|i\rangle\}_{i=1}^N$ being a basis for $\mathcal{H}$, and $|\sigma|$ denoting the parity of σ. Now we observe that, since $|\psi_A\rangle$ is a $\mathrm{sl}(r + 1)$ *singlet*, one must have $H_\alpha|\psi_A\rangle = E_\alpha|\psi\rangle = E_{-\alpha}|\psi\rangle = 0$ ($\alpha = 1, \ldots, r$). Therefore for $|\psi_B\rangle$ any vector of $\mathcal{H}_B$, $|\psi_A\rangle \otimes |\psi_B\rangle$ is annihilated by the interaction Hamiltonian and is an eigenstate of $H_S + H_B$ if $|\psi_B\rangle$ is an eigenstate of H_B. More generally for $N = m(r + 1), (m \in \mathbf{N})$, one has the $(r+1, m)$ Young diagram with multiplicity $n(N)$, still corresponding to one-dimensional representations of $\mathrm{sl}(r+1)$. Let $|\psi_j^{(N)}\rangle$, $[j = 1, \ldots, n(N)]$ denote the associated vectors, then, reasoning as above, we have $|\psi_j^{(N)}\rangle \otimes |K\rangle_B$ as an eigenstate of H_{SB} with eigenvalue $E_K = \sum_j \omega_{k_j}$. With the procedure described above, we have therefore built an infinite family of *exact* eigenstates of the interacting Hamiltonian H_{SB} that is given by simple tensor products. This allows us to state straightforwardly the following generalization of Theorem 1:

Theorem 2.—Let $C_N = \mathrm{span}\{|\psi_j^{(N)}\rangle | j = 1, \ldots, n(N)\}$, with $N = 0 \bmod (r+1)$, and $\mathcal{M}_N$ the manifold of the states over C_N. Then, if $\rho \in \mathcal{M}_N$, for any state ρ_B over $\mathcal{H}_B$ one has $L_t^{\rho_B} \rho = \rho$.

Proof.—The proof proceeds as in the qubit case. The code is nothing but C_N itself. For $N = 2(r+1)$, one has $n(N) = 2$ and a single *qubit* can be encoded. As far as the encoding efficiency is concerned, we observe that, in the $r = 1$ case, one has $n(N) = N![(N/2)!(N/2+1)!]^{-1}$ (N even) from which follows, for large N, the asymptotic form $\log_2 n(N) \simeq N - 3/2 \log_2 N$. The latter equation tells us that, for a large replica number, one has an encoding efficiency $N^{-1} \log_2 n(N)$ approximately of one qubit per replica, whereas the fraction $2^{-N} n(N)$ of the Hilbert space occupied by the code is vanishingly small. In the general case, $r > 1$, the multiplicities $n(N)$ are the Littlewood-Richardson coefficients [9].

A few important remarks extending theorem 2 follow. (i) When only the dephasing terms are present, due to the fact that the resulting model can be diagonalized by a unitary transformation in each $\mathcal{A}_S$-weight space [10], if ρ is a state over $\mathcal{H}_S(\lambda)$, then $L_t^{\rho_B} \rho = \rho$. This latter result, in its simplest form (i.e., $r = 1$), can be found in [11] and [12]. Notice that this model does not take into account the amplitude errors induced by the bath. (ii) We can allow also for interactions H_{SS} between replicas, provided they leave C_N invariant. For example, it would be sufficient that $\mathcal{A}_S$ were a symmetry algebra for H_{SS}. There results $L_t^{\rho_B} \rho = U_S(t) \rho U_S^\dagger(t)$, where $U_S(t) = e^{-iH_{\mathrm{SS}} t}$; therefore the Liouvillian dynamics is still unitary but no longer trivial. (iii) Since C_N is an irreducible S_N representation space, the theorem still holds if the Hamiltonian H_S and the system operators coupled with the bath belong to the symmetric subspace of $\mathrm{End}(\mathcal{H}_S)$. From the physical point of view this means that we can allow for replica-replica and replica-bath interactions involving many excitations (powers of the e_α^i's), provided that all the replicas are treated symmetrically.

We expect that if the key assumption of a replica-symmetric coupling with the bath is slightly violated—for example, the system is coupled with modes with wavelengths shorter than the inter-replica distance—the proposed encodings have a low error rate, in analogy with the "subdecoherent" states in [11].

In summary, we have shown that, for open quantum systems, made of N replicas of a given system S, coupled with a common environment in a replica-symmetric fashion, one can build—for sufficiently large N—a subspace C_N of $\mathcal{H}^{\otimes N}$ that does not get entangled with the environment. The whole class of (possibly nonlinear) replica-replica interactions which leave C_N invariant together with the replica-symmetric system-bath interactions (which possibly annihilate C_N) is consistent with this scheme. Such subspace is nothing but the singlet sector of the dynamical algebra $\mathcal{A}_S$ of S, the direct sum of the one-dimensional representations of $\mathcal{A}_S$. This elegant result allows us, in principle, to design noiseless (i.e., dissipation-decoherence-free) quantum codes. From the point of view of the practical implementation, the difficulties one may expect to face with these codes depend on the limitations inherent with the code words preparation and on the large bath coherence length required. The question of the code stability, in the case in which the latter requirement is not satisfied, can be addressed in the framework of the Liouville–von Neumann equation formalism [13]. Another open question is whether the approach discussed may possibly be extended to the case when $\mathcal{H}$ is infinite dimensional. Work is in progress along these lines.

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