Subharmonic Entrainment of Unstable Period Orbits and Generalized Synchronization

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Chaos synchronization of nonidentical unidirectionally coupled systems is investigated and characterized in terms of the entrainment ratios of unstable periodic orbits. The implications of subharmonic entrainment for generalized synchronization are discussed and illustrated for discrete and continuous systems. [S0031-9007(97)04380-9]

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Synchronization phenomena are of fundamental importance for many physical, biological, and technical systems. In particular, the synchronization of chaotic dynamics [1] has attracted much attention during the last years because of its role in understanding the basic features of coupled nonlinear systems and in view of potential applications in communication systems and time series analysis and modeling [2]. Different coupling schemes have been proposed in order to achieve synchronization, in particular, for unidirectionally coupled systems [1] including spatially extended systems like coupled oscillators or partial differential equations [3]. If both coupled systems are of the same type *identical synchronization* may occur where the states $\mathbf{x}$ and $\mathbf{y}$ of drive and response, respectively, converge to the same values (i.e., $\|\mathbf{x}(t) - \mathbf{y}(t)\| \to 0$ for $t \to \infty$). If the coupled systems are different (for example, due to parameter mismatch) more sophisticated types of synchronization like *generalized synchronization* [4–10], *phase synchronization* [11], or *lag synchronization* [12] have been observed, and the variety and complexity of these synchronization phenomena is currently investigated very intensely.

The goal of this Letter is to draw attention to the synchronization features of the *unstable periodic orbits* (UPOs) that constitute the skeleton of any typical chaotic attractor. Only recently, these orbits have been identified to be a possible source of intermittent breakdown of synchronization in the presence of noise or other perturbations [13]. For robust high-quality synchronization any UPO of the drive has to generate a stable synchronous periodic orbit (PO) of the response system. If, for example, due to a too weak coupling, this condition is not fulfilled the synchronization breaks down when the driving chaotic trajectory comes sufficiently close to an UPO that fails to entrain a stable PO of the response system, and (arbitrary) small amounts of noise may generate large scale deviations from the synchronized state. Often such unstable POs of the response system are the result of a period doubling bifurcation where a formerly stable periodic response orbit becomes unstable and a new period-2 orbit is created which is not located in the set of synchronized motion $M = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} = \mathbf{y}\}$ (in the case of identical synchronization). Therefore, high-quality synchronization is closely connected with subharmonic response that may be characterized by its *entrainment ratio* $T_D : T_R = 1 : p$, where T_D and T_R denote the periods of the drive UPO and the response PO, respectively, and p is an integer. However, *subharmonic entrainment* with $p > 1$ not only has important consequences for high-quality synchronization but also for generalized synchronization and predictability. This is the main topic of this paper.

In the following, we shall consider different types of *generalized synchronization* (GS). Two unidirectionally coupled systems X and Y may be called *synchronized* if the behavior of the response system Y is completely determined by the drive system X. More precisely, GS of unidirectionally coupled dynamical systems $\dot{\mathbf{x}} = f(\mathbf{x})$ ($\mathbf{x} \in R^m$) and $\dot{\mathbf{y}} = g(\mathbf{y}, \mathbf{x})$ ($\mathbf{y} \in R^k$) occurs if there exists an open *synchronization basin* $B \subset R^m \times R^k$ such that $\lim_{t\to\infty} \|\mathbf{y}(t; \mathbf{x}_0, \mathbf{y}_{10}) - \mathbf{y}(t; \mathbf{x}_0, \mathbf{y}_{20})\| = 0$ for all $(\mathbf{x}_0, \mathbf{y}_{10}), (\mathbf{x}_0, \mathbf{y}_{20}) \in B$, i.e., if the response system Y is asymptotically stable with respect to the drive X [14]. This is not the only possible definition. In the first work on general types of synchronous chaotic dynamics by Afraimovich *et al.* [4] a stronger notion of GS was introduced assuming essentially the existence of a homeomorphism between states of drive and response. A similar definition was used in Refs. [5,7,9] where GS was defined in terms of a (not necessarily homeomorphic) function $H : R^m \to R^k$ such that $\lim_{t\to\infty} \|H(\mathbf{x}(t)) - \mathbf{y}(t)\| = 0$. Methods for investigating this type of GS using time series have been suggested in Ref. [5]. In Ref. [7] it was argued that GS yielding a functional relation occurs for invertible drive systems if the response system is asymptotically stable. This conclusion holds for aperiodic orbits and for (entrained) periodic oscillations with $T_D = T_R$, but *not* if $T_D : T_R = 1 : p$ where $p > 1$ [8,15]. A simple example is the case of periodic synchronization with ratio $T_D : T_R = 1 : 2$ where any point on the attractor of the drive is mapped to *two* points on the response

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orbit. H is in this case a relation but *not* a function [16]. In the following we discuss the relevance of this multivaluedness of H for chaotic systems.

As our first example we consider the *generalized baker map* [9,17],

$$x_1^{n+1} = \begin{cases} \lambda_a x_1^n, & \text{if } x_2^n < a, \\ \lambda_a + \lambda_b x_1^n, & \text{if } x_2^n \geq a, \end{cases} \quad x_2^{n+1} = \begin{cases} x_2^n/a, & \text{if } x_2^n < a, \\ (x_2^n - a)/b, & \text{if } x_2^n \geq a, \end{cases} \tag{1}$$

that drives the one dimensional system

$$y^{n+1} = \arctan(-c * y^n) + x_1^{n+1} + d. \tag{2}$$

The parameters of the driving system are $\lambda_a = 0.15$, $\lambda_b = 1 - \lambda_a$, $a = 0.1$, $b = 1 - a$, and for the response system we use $c = 40$ and $d = 1$. To illustrate the relation between the states $\mathbf{x}^n = (x_1^n, x_2^n)$ and y^n of the drive and the response, respectively, we have plotted in Fig. 1 y^n vs x_1^n. As can be seen in this figure the graph of (x_1, y) consists essentially of two branches [18]. In order to understand the features of this relation between drive and response we have investigated the entrainment properties of some low-period UPOs of the drive. The generalized baker map (1) possesses two fixed points at $\mathbf{x} = (0,0)$ and $\mathbf{x} = (1,1)$. If the first fixed point $(0,0)$ is used to drive the response system (2) the response orbit $\{y^n\}$ consists of an alternating sequence $\{\ldots, -0.5609, 2.5262, -0.5609, 2.5262, \ldots\}$ while the response to the second fixed point $(1,1)$ is constant with $y^n \approx 0.4811$. Thus one of the fixed points of the driving system is mapped to a fixed point of the response, and the other fixed point of the drive is related to a period two orbit of the response system. A different situation occurs for the period-2 orbit UPO of the drive (1)

$$\mathbf{x}^n = \left(\frac{\lambda_a}{1 - \lambda_a \lambda_b}, \frac{a^2}{1 - ab}\right) \approx (0.171\,920, 0.010\,989\,0),$$

$$\mathbf{x}^{n+1} = \left(\frac{\lambda_a^2}{1 - \lambda_a \lambda_b}, \frac{a}{1 - ab}\right) \approx (0.025\,788, 0.109\,890),$$

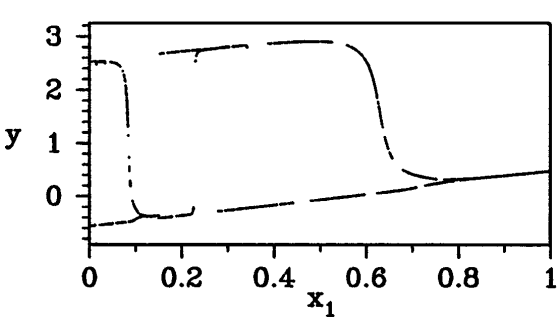

FIG. 1. Variable y of the response system (2) vs variable x_1 of the drive system (1).

for which two basins of synchronization ($y < y_b$ and $y > y_b$, $y_b \approx 0.051\,41$) occur that both lead to $1:1$ entrainment of the response system with $\mathbf{x}^n \mapsto y^n \approx 2.696\,09$, $\mathbf{x}^{n+1} \mapsto y^{n+1} \approx -0.535\,736$ for initial values $y < y_b$ and $\mathbf{x}^n \mapsto y^n \approx -0.389\,005$, $\mathbf{x}^{n+1} \mapsto y^{n+1} \approx 2.532\,41$ for initial values $y > y_b$. In both cases the two points of the driving period-2 UPO are mapped uniquely to two different points of the response system. Furthermore, it turned out that the two basins are only relevant for this particular period-2 orbit but not for the chaotic dynamics [i.e., two orbits starting in different basins finally converge to the same trajectory when driven by a chaotic sequence from (1)].

Both period-3 UPOs of (1) generate period-6 orbits of the response system, and thus H is again not a function for these UPOs. Similar relations occur for other higher periodic orbits and may be interpreted as the reasons for the rather complicated nonsmooth dependence of y^n on x_1^n (or $\mathbf{x}^n$). Since the Lyapunov exponent of the response system $\lambda^R \approx -2.152$ is smaller than the smallest exponent of the drive $\lambda_2^D \approx -0.336$, this example also shows that a preservation of the Lyapunov dimension is not sufficient to guarantee the existence of a smooth function H [10]. In Ref. [9] it was shown that such a condition for so-called past-history Lyapunov exponents has to be fulfilled for *all* points $\mathbf{x}$ on the drive attractor in order to have a smooth map [or *differentiable* GS (DGS)]. Therefore, we have checked the past Lyapunov exponents of all UPOs (up to period 10), but in all these cases the smallest exponent of the drive was larger than the exponent of the response system. Thus stability criteria seem not to be sufficient for smoothness in cases where subharmonic entrainment occurs and H is not a function for all $\mathbf{x}$.

Similar subharmonic entrainment phenomena can be observed with continuous dynamical systems as will be illustrated now for two coupled Rössler systems,

$$\begin{aligned} \dot{x}_1 &= 2 + x_1(x_2 - 4), & \alpha \dot{y}_1 &= 2 + y_1(y_2 - 4), \\ \dot{x}_2 &= -x_1 - x_3, & \alpha \dot{y}_2 &= -y_1 - y_3, \\ \dot{x}_3 &= x_2 + b x_3, & \alpha \dot{y}_3 &= y_2 + b y_3 + c(x_3 - y_3). \end{aligned} \tag{3}$$

The parameter $\alpha = 2$ is used to detune both systems and c is a variable coupling constant. Figure 2(a) shows a projection of the driving Rössler attractor (gray solid curve) together with two UPOs (black solid and dashed curves) that are embedded in the chaotic attractor. In Fig. 2(b) the corresponding chaotic and periodic orbits of the response system are shown. Since the detuning parameter $\alpha = 2$ is different from 1 both systems cannot synchronize identically (i.e., with $\|\mathbf{x}(t) - \mathbf{y}(t)\| \to 0$ for $t \to \infty$). The coupling constant $c = 0.27$ has been chosen such that (i) the largest conditional Lyapunov exponent of the response system is negative and (ii) different entrainment ratios for different pairs of UPOs occur. The UPO plotted as a solid curve in Fig. 2(a) locks with a periodic response of the same period [see Fig. 2(c)],

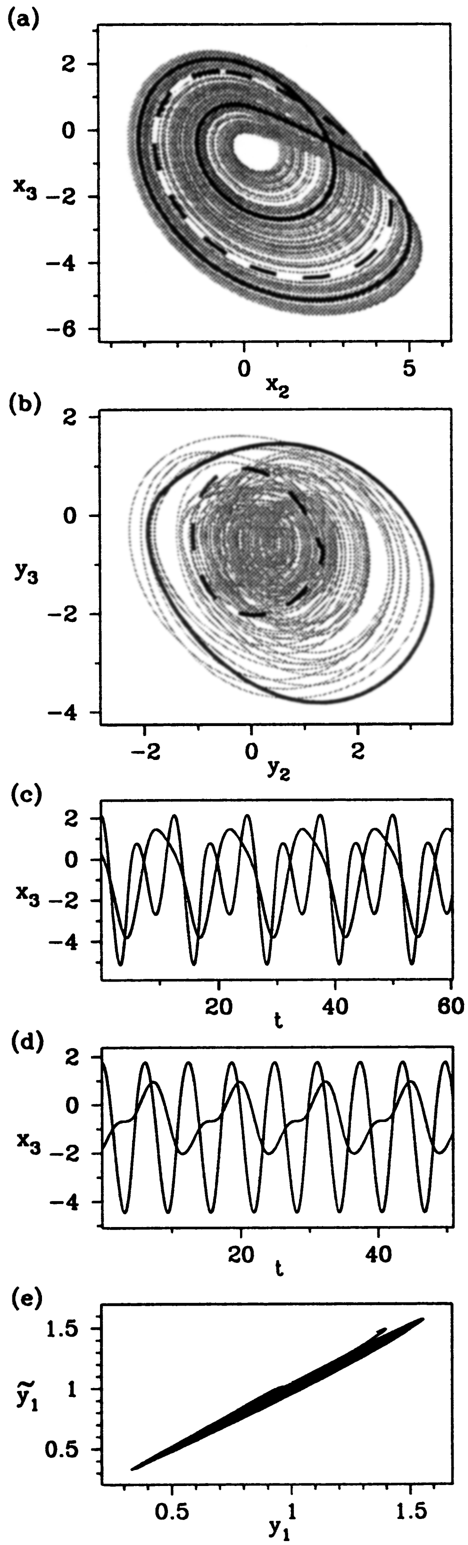

FIG. 2. Coupled Rössler systems (3) for $c = 0.27$ and $b = 0.42$. (a) Chaotic drive attractor (solid gray curve) and two UPOs (solid and dashed black curves). (b) Response attractor (gray) and POs (solid and dashed black curves) that are entrained by the UPOs shown in (a). (c) Periodic oscillation of the first UPO and its corresponding PO (1 : 1). (d) Same as (c) for the second UPO-PO pair (1 : 2). (e) y_1 variables of two slightly different response systems.

whereas the dashed UPO in Fig. 2(a) leads to a period-2 solution when used to drive the response system [see Fig. 2(d)]. Because of this 1:2 entrainment any point on the UPO of the drive system corresponds to two points on the response orbit.

Nevertheless, drive and response are synchronized in the sense of the first definition given above. This is illustrated in Fig. 2(e) where the y_1 variables of two response systems (3) with different initial conditions are plotted on the ordinate and abscissa, respectively [8]. To estimate the robustness of their mutual synchronization the parameters $b = 0.422$ and $b = 0.418$ of the response systems have been chosen slightly different from the corresponding value $b = 0.42$ of the drive in Eq. (3). Because of this parameter mismatch the motion of both response systems is not located exactly on the diagonal in Fig. 2(e) but always remains very close to it without large excursions which would be a typical signature for any transversal instabilities [13]. Similar results have been obtained when noise was added to the coupling signal.

If the coupling constant c is increased all UPOs we have found entrain with $T_D = T_R$, but there still exist regions on the drive attractor where nearest neighbors of some reference state are mapped to different branches on the response attractor. This phenomenon is illustrated in Fig. 3 which was generated in the following way: (i) Select at time t a state $\mathbf{x}(t)$ of the drive and its $nn = 10$ nearest neighbors, (ii) compute the distances of the corresponding states of the response system from the state $\mathbf{y}(t)$, and (iii) plot these distances vs the time t. As can be seen in Fig. 3, the trajectory enters different regions on the attractor, where a neighborhood of the current drive state is mapped to a different number of locations (clusters) on the response attractor. In the case of DGS [9] only a single cluster with (very) small distances should occur which is not the case for this example. This observation is in agreement with the fact that the largest Lyapunov exponent of the response system $\lambda_1^R = -0.44$ is not smaller than the smallest LE of the drive $\lambda_3^D = -3.2$ and thus a function H may exist which is not smooth. This method for visualizing the relation between drive and response can also be applied to experimental data

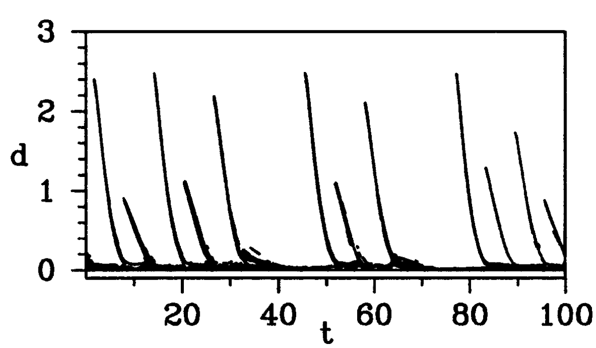

FIG. 3. Distances d of response states that correspond to 10 nearest neighbors of the drive system ($c = 2$, $b = 0.42$).

in a reconstructed state space provided that the time series is sufficiently long (or the relation between drive and response is sufficiently smooth).

In this Letter we have shown that UPOs which are embedded in a chaotic drive attractor may generate periodic orbits of the response system with different entrainment ratios $T_D : T_R = 1 : p$. Subharmonic entrainment with $p > 1$ means that there exist points on the drive attractor which are not mapped uniquely to the response attractor. If generalized synchronization is defined as the existence of an asymptotic function $\mathbf{y} = H(\mathbf{x})$ that is valid on the whole drive attractor (including all UPOs, etc.), then the concept of GS does not apply as soon as an UPO occurs that entrains a PO of the response with a longer period as that of the drive. On the other hand, we have demonstrated that even in such cases a pair of slightly different response systems synchronizes robustly. Therefore, we suggest to distinguish two types of generalized synchronization: (i) where different response trajectories $\mathbf{y}(t)$ and $\tilde{\mathbf{y}}(t)$ starting in some common basin yield asymptotically the same dynamics ($\lim_{t\to\infty} \|\mathbf{y}(t) - \tilde{\mathbf{y}}(t)\| = 0$) or (ii) where a function H exists with $\lim_{t\to\infty} \|H(\mathbf{x}(t)) - \mathbf{y}(t)\| = 0$. The second definition was used in Refs. [5,7,9] and in a stronger form in [4]. It applies, for example, to those cases where a global Lyapunov function for the response system exists (e.g., the first example in Ref. [7]) and subharmonic entrainment and the resulting complications for H can be excluded. The first definition is closely related to the auxiliary system method of Abarbanel *et al.* [8] and more general, because it covers also subharmonic entrainment. However, it is probably still not the most general definition, because it does not include cases of partial synchronization like phase synchronization [11]. The development of useful and rigorous definitions for the full hierarchy of synchronized motion (including noise) thus remains an important task for future investigations.

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