Solution of the Odderon Problem for Arbitrary Conformal Weights

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A new method is applied to solve the Baxter equation for three coupled, noncompact spins. Because of the equivalence with the system of three Reggeized gluons, the intercept of the odderon trajectory is predicted for the first time, as the analytic function of the two relevant parameters. [S0031-9007(97)04273-7]

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Calculation of the QCD prediction for the intercept of the odderon trajectory still remains a challenge for the leading logarithmic scheme of the Reggeization of QCD [1,2]. In the first approximation the problem naturally separates into sectors with fixed number n of the Reggeized gluons propagating in the t channel. The lowest nontrivial case, $n = 2$, was solved in the classical papers by Balitskii, Fadin, Kuraev, and Lipatov (BFKL) [3] resulting in the simple expression for the intercept of the hard pomeron. The notable progress for arbitrary n was achieved by Lipatov, and Faddeev and Korchemsky [4,5] who have established exact equivalence with the one dimensional chain of n noncompact spins. Leading high energy behavior of QCD amplitudes is given by the highest eigenvalue of the corresponding Heisenberg Hamiltonian of n spins with nearest-neighbor interaction. Moreover, by identifying enough constants of motion they were able to prove that this system is soluble for arbitrary n. The success of this, rather mathematical, approach was confirmed by rederiving the Lipatov *et al.* result in the $n = 2$ case [5,6]. However, the adopted procedure requires an analytic continuation from the integer values of the relevant conformal weight h (see later) because only for integer h they were able to diagonalize the two-spin Hamiltonian. The $n = 3$ case, which gives the lowest contribution to the odderon exchange, was also studied by Faddeev and Korchemsky, and Korchemsky [5,6]. Again, the spectrum of the system for integer h can be found for any finite $h = m$. However, the general expression for arbitrary m is not known, and consequently the analytical continuation to $h = 1/2$ is not available. (The lowest state of the $n = 3$ Hamiltonian is believed to occur at $h = 1/2$.)

We have developed a new approach which (a) works for arbitrary values of the conformal weight h, providing explicitly above continuation, and (b) gives the analytic solution of the $n = 3$ case for arbitrary h and q_3. For $n = 2$ our method reproduces again the BFKL result clarifying the problem of boundary conditions for arbitrary h. This and the details of the $n = 3$ calculation will be presented elsewhere [7]. In this Letter we address directly the odderon case. We rely on the results derived in Refs. [4–6] and follow the conventions and notation of Refs. [5,6].

The intercept of the odderon trajectory is given by

$$\alpha_O(0) = 1 + \frac{\alpha_s N_c}{4\pi}\left[\epsilon_3(h, q_3) + \overline{\epsilon}_3(\overline{h}, \overline{q}_3)\right], \quad (1)$$

where ϵ_3 and $\overline{\epsilon}_3$ are, respectively, the largest eigenvalues of the $n = 3$ Reggeon Hamiltonian and its antiholomorphic counterpart [5,6]. This system is equivalent to the one dimensional chain of three noncompact spins with nearest-neighbor interactions. Applying Bethe ansatz to the latter one obtains

$$\epsilon_3 = i\left(\frac{\dot{Q}_3(-i)}{Q_3(-i)} - \frac{\dot{Q}_3(i)}{Q_3(i)}\right) - 6, \quad (2)$$

where $Q_3(\lambda)$ satisfies the following Baxter equation:

$$(\lambda + i)^3 Q_3(\lambda + i) + (\lambda - i)^3 Q_3(\lambda - i) = (2\lambda^3 + q_2\lambda + q_3) Q_3(\lambda). \quad (3)$$

q_2 and q_3 are the eigenvalues of the two, commuting with the Hamiltonian, operators which play an important role in the proof of the solubility of the above system [4,5]. The spectrum of $\hat{q}_2$ is known from the symmetry considerations

$$q_2 = h(1 - h), \qquad h = \tfrac{1}{2}(1 + m) - i\nu, \qquad m \in Z, \qquad \nu \in R. \quad (4)$$

The eigenvalues of $\hat{q}_3$ are known only for integer conformal weights h, whereas the value of q_3 for the ground state of the three Reggeized gluons ($h = 1/2$) is not available. Analogous expressions hold for the antiholomorphic sector with $\overline{h} = (1 - m)/2 - i\nu$ [6].

Our goal is to determine $\epsilon_3(h, q_3)$ for arbitrary h and q_3. To this end we begin with the trick of Ref. [8] and seek the solution of the Baxter equation (3) in the form of the *double* contour representation

$$Q_3(\lambda) = \int_{C_I} Q_I(z) z^{-i\lambda-1}(z - 1)^{i\lambda-1} dz + \int_{C_{II}} Q_{II}(z) z^{-i\lambda-1}(z - 1)^{i\lambda-1} dz. \quad (5)$$

Provided the boundary contributions cancel, Eq. (3) is equivalent to the following ordinary differential equation for $Q_I(z)$ and $Q_{II}(z)$ [$\equiv Q(z)$]

$$\left[\left(z(1-z)\frac{d}{dz}\right)^3 - q_2 z^2(1-z)^2\frac{d}{dz} + iq_3 z(1-z)\right]Q(z) = 0. \quad (6)$$

This is a third order linear equation of the Fuchs class with the three regular singular points at $z = 0, 1$, and ∞, considered earlier in Refs. [5,6,9,10]. We will prove that the complete boundary conditions on $Q_I(z)$ and $Q_{II}(z)$ are *uniquely* determined by the requirement of the cancellation of the boundary terms among the integrals (5). This is the distinctive feature of the $n = 3$ case which allows for the successful application of our strategy.

We begin with the construction of the two fundamental sets of three, linearly independent solutions

$$\begin{aligned} \vec{u}(z) &= (u_1(z), u_2(z), u_3(z)), \\ \vec{v}(z) &= (v_1(z), v_2(z), v_3(z)), \end{aligned} \quad (7)$$

around $z = 0$ and $z = 1$, respectively.

$$\begin{aligned} u_1(z) &= \sum_{n=0}^{\infty} f_n z^n, \\ u_2(z) &= \frac{1}{\pi i}\ln z u_1(z) + \frac{1}{\pi i}\sum_{n=0}^{\infty} r_n^{(1)} z^n, \\ u_3(z) &= \frac{1}{\pi^2}\ln^2 z u_1(z) + \frac{2}{\pi^2}\ln z \sum_{n=0}^{\infty} r_n^{(1)} z^n \\ &\quad + \frac{1}{\pi^2}\sum_{n=0}^{\infty} r_n^{(2)} z^n, \end{aligned} \quad (8)$$

where the coefficients of the expansions are determined by the recursion relations easily obtained from Eq. (6).

$$\begin{aligned} f_{n+1} &= (b_n f_n - c_{n-1} f_{n-1})/a_{n+1}, \\ f_0 &= 1, \qquad f_{-1} = 0, \\ a_n &= n^3, \\ b_n &= iq_3 + n(q_2 + (2n+1)(n+1)), \\ c_n &= n(q_2 + (n+1)(n+2)), \end{aligned} \quad (9)$$

and for the logarithmic solutions

$$\begin{aligned} r_{n+1}^{(1)} &= (-p_n^{(1)} + b_n r_n^{(1)} - c_{n-1} r_{n-1}^{(1)})/a_{n+1}, \\ r_0^{(1)} &= 1, \qquad r_{-1}^{(1)} = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} p_n^{(1)} &= 3(n+1)^2 f_{n+1} - [1 + q_2 + 6n(n+1)]f_n \\ &\quad + (-1 + q_2 + 3n^2) f_{n-1}, \end{aligned} \quad (11)$$

$$\begin{aligned} r_{n+1}^{(2)} &= (-p_n^{(2)} + b_n r_n^{(2)} - c_{n-1} r_{n-1}^{(2)})/a_{n+1}, \\ r_0^{(2)} &= 1, \qquad r_{-1}^{(2)} = 0, \end{aligned} \quad (12)$$

$$\begin{aligned} p_n^{(2)} &= 6(n+1)f_{n+1} - 6(2n+1)f_n + 6nf_{n-1} \\ &\quad + 6(n+1)^2 r_{n+1}^{(1)} - 2[1 + q_2 + 6n(n+1)]r_n^{(1)} \\ &\quad + 2[2 + q_2 + 3(n^2-1)]r_{n-1}^{(1)}. \end{aligned} \quad (13)$$

The series in Eq. (8) are convergent in the unit circle K_0 around $z = 0$. Similarly one can construct the $\vec{v}(z)$ solutions in the unit circle K_1 around $z = 1$. In fact, because of the symmetry of the Eq. (6) we take

$$\vec{v}(z; q_2, q_3) = \vec{u}(1 - z; q_2, -q_3). \quad (14)$$

Since any solution is the linear combination of the fundamental solutions, we have

$$\begin{aligned} Q_I(z) &= au_1(z) + bu_2(z) + cu_3(z) \\ &\equiv A \cdot \vec{u}(z) = A \cdot \Omega\vec{v}(z), \\ Q_{II}(z) &= du_1(z) + eu_2(z) + fu_3(z) \\ &\equiv B \cdot \vec{u}(z) = B \cdot \Omega\vec{v}(z), \end{aligned} \quad (15)$$

with an obvious vector notation. The transition matrix Ω is defined by

$$\vec{u}(z) = \Omega\vec{v}(z), \quad (16)$$

and plays an important role in the following. It provides the analytic continuation of our solutions $Q(z)$ between K_0 and K_1. Transition matrix Ω can be easily determined from Eq. (16) once both bases, Eqs. (8) and (14), are known. For example, the ith row, $\vec{\omega}_i^T$, of Ω can be obtained as

$$\begin{aligned} \vec{\omega}_i &= (\Sigma^T)^{-1}\vec{w}_i, \qquad \Sigma_{kr} = v_k(z_r), \\ (\vec{w}_i)_r &= u_i(z_r), \qquad i, k, r = 1, 2, 3, \end{aligned} \quad (17)$$

where z_1, z_2, and z_3 are arbitrary three points inside the intersection of K_0 and K_1. Next we introduce the monodromy matrix M which describes the behavior of the basis $\vec{u}$ upon the 2π rotation around the branch point $z = 0$

$$\vec{u}(z_{\mathrm{end}}) = M\vec{u}(z_{\mathrm{start}}), \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -4 & 1 \end{pmatrix}. \quad (18)$$

We are now ready to write the condition for the cancellation of the boundary contributions in Eq. (5). Let us choose the contours C_I and C_{II} as shown in Fig. 1. Then, the boundary contributions cancel if

$$A^T M_I = P^T, \qquad B^T M_{II} = -P^T, \qquad P^T = (\alpha, \beta, \gamma), \quad (19)$$

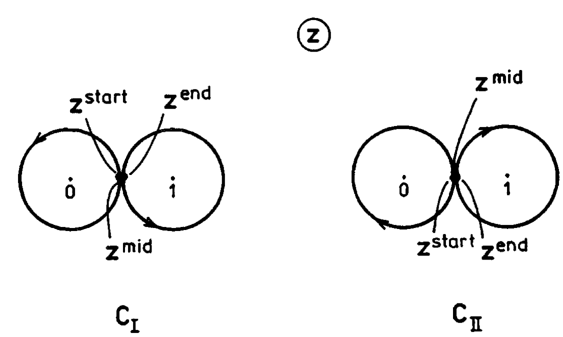

FIG. 1. Integration contours used in Eq. (5). Start z^{start}, middle z^{mid}, and end z^{end} points coincide, but they lay on the different sheets of the Riemann surface of the integrands.

where the combined monodromy matrices for the corresponding contours read

$$M_I = \Omega M \Omega^{-1} - M^{-1}, \qquad M_{II} = \Omega M^{-1} \Omega^{-1} - M. \tag{20}$$

(Matching the coefficients of expansion into the fundamental set of solutions guarantees *functional* equality, i.e., including derivatives as well.)

Hence the original freedom of six coefficients in Eqs. (15) was reduced to the three free parameters which we conveniently choose as α, β, and γ. This was expected. However, in the $n = 3$ case additional simplification occurs which, remarkably, allows one to fix completely the remaining freedom.

The key point is the observation that the monodromy matrices (20) are singular, i.e., $\det(M_I) = \det(M_{II}) = 0$. To see this it is enough to inspect the Riemann P symbol corresponding to Eq. (6).

$$P \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & 0 \\ 0 & 0 & 1+h \\ 0 & 0 & 2-h \end{matrix} ; z \right\}. \tag{21}$$

It is readily seen that, contrary to the $n = 2$ case, there exists a solution which is *regular* at $z = \infty$. Therefore the reduced monodromy matrices M_I and M_{II} must have the zero eigenvalue corresponding to this solution. As a consequence, Eqs. (19) do not have the unique solution for $(a, \ldots, f)$. Different choices of coefficients differ by the zero mode. This difference is inessential because the integrals of the solution, regular at infinity, vanish. We therefore proceed to isolate the zero mode explicitly, and then impose the condition of cancellation of boundary contributions. To this end we introduce the new basis $\vec{t}(z)$ such that $t_3(z)$ is the solution regular at $z = \infty$. Transformation matrix $\mathcal{T}$, $\vec{u}(z) = \mathcal{T}\vec{t}(z)$, to the $\vec{t}(z)$ basis can be readily obtained by diagonalizing the commuting matrices $M^A = MM_I$ and $M^B = M_{II}M^{-1}$. In the new basis (marked by the subscript t) the cancellation condition reads

$$A_t'^T M_t^A = P_t^T, \qquad B_t^T M_t^B = -P_t^T M_t^{-1}, \tag{22}$$

where M_t^A and M_t^B are diagonal with one (third, say) zero eigenvalue, $A_t'^T \equiv (a_t', b_t', c_t') = A_t^T M_t^{-1}$, and $M_t = \mathcal{T}^{-1} M \mathcal{T}$ is the monodromy matrix in the new basis. Our final conclusions follow now trivially from Eq. (22). Coefficients c_t' and f_t are arbitrary (and irrelevant), $\gamma_t = 0$ and a_t', b_t', d_t, e_t are determined uniquely by α_t and β_t provided the *additional consistency condition,*

$$\alpha_t m_{13} + \beta_t m_{23} = 0, \tag{23}$$

is satisfied, m_{ik} being the matrix elements of M_t^{-1}. The last condition fixes completely the remaining freedom. Now the integral transforms $Q_I(z)$ and $Q_{II}(z)$ [hence also the solution of the Baxter equation $Q(\lambda)$] are determined uniquely up to an irrelevant normalization. This ends our proof.

Now the contour integrals and derivatives over λ can be done analytically since C_I and C_{II} lay within the corresponding domains of convergence of all involved series [we use the $u(v)$ basis on $C_I(C_{II})$]. Integrating resulting expressions term by term [consistent choice of the appropriate branches of the kernel and of the multivalued functions $Q_{I,II}(z)$ must be made [7]] we have obtained the final formula for $\epsilon_3(h, q_3)$ in the form of absolutely convergent series for arbitrary values of the conformal weight h and q_3. The resulting expression is rather lengthy and will not be quoted here; however, it provides, for the first time, the energy of the three Reggeon Hamiltonian as the analytic function of the relevant parameters.

Let us discuss now some consistency checks of our result. First, for integer $h = m$ there exists a discrete set of quantized values of $q_3 = q_3^k(m)$ for which the polynomial solution $u_1(z)$ exists. This quantization of q_3 is known, and the eigenenergies $\epsilon_3(\overline{\epsilon_3})$ at these points can be calculated [6]. We quote the first few levels in Table I.

Our formula reproduces these results exactly providing continuous interpolation between values quoted in Table I. In fact, at these "polynomial values" of h and q_3 our expression contains undefined terms of the type $\infty \times 0$. However, the limit $(h, q_3) \to (m, q_3^k(m))$ is finite and gives levels in Table I.

Second, our expression agrees with the asymptotic formula derived in Ref. [11]. In the limit $h, q_3 \to \infty$, $q_3/h^3 = \text{const}$, Korchemsky has derived a simple expression

$$\epsilon_3(h, q_3) = -2\ln 2 - \sum_{k=0}^{3} [\psi(1 + ihx_k) + \psi(1 - ihx_k) - 2\psi(1)], \tag{24}$$

where x_k, $k = 1, 2, 3$ are three roots of the polynomial $2h^3x^3 + h^2(1 - h)x + q_3$. Figure 2 shows comparison of this asymptotic form with our exact formula for $q_3/h^3 = 1$. Agreement is very striking indeed and persists to h as low as $h \sim 0.4$. Interestingly, it turns out that the expression (24) contains many terms which are nonleading in the above limit. Retaining consistently the terms up to a given order in $1/h$, as was also done in Ref. [11], gives yet a simpler result which, however, does not work so well. Also the analytic structure of $\epsilon_3(h, q_3)$ is reasonably well reproduced by Eq. (24) while the rigorous expansion in $1/h$ fails here (see later).

TABLE I. Quantization of q_3 and first few levels of the holomorphic Hamiltonian in the polynomial case.

h	q_3	ϵ_3
4	$\pm 2\sqrt{3}$	$-7\frac{1}{2}$
5	$\pm 6\sqrt{3}$	$-8\frac{5}{6}$
6	$\pm 4\sqrt{3}$ $\pm 4\sqrt{30}$	$-9\frac{1}{4}$ -10

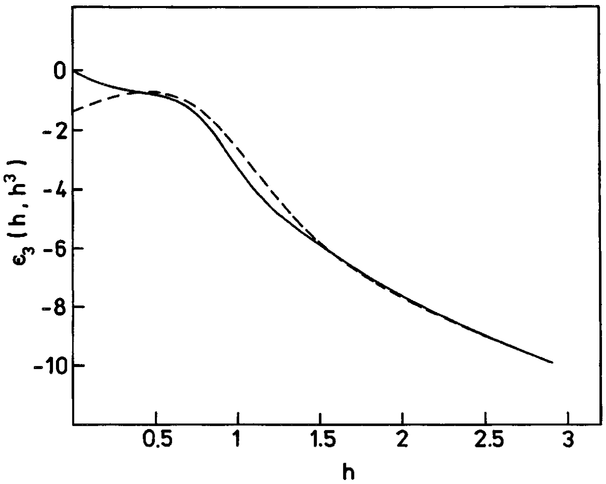

FIG. 2. Comparison of the exact result (solid line) with the asymptotic formula (dashed line), Eq. (24), described in text.

Finally, we discuss some consequences and relations of this result with other works. Since the lowest state of the three Reggeized gluons is expected to occur at $h = \frac{1}{2}$, we have mapped numerically the analytic structure of $\epsilon_3(\frac{1}{2}, q_3)$ in the complex q_3 plane. It turns out that the holomorphic energy $\epsilon_3(\frac{1}{2}, q_3)$ has a series of simple poles on the imaginary axis, and behaves regularly in the remaining part of the q_3 plane. In fact, Re $\epsilon_3(\frac{1}{2}, q_3)$ is negative almost in the whole q_3 plane except for the small regions in the vicinity of the above poles. Interestingly, the approximate solution, Eq. (24), has the same singularity structure with similar location of poles. This suggests that there may exist a better approximation scheme than $1/h$ expansion in which Eq. (24) is the lowest order.

Because of Eq. (1) and the symmetry ($\overline{\epsilon}_n = \epsilon_n^\star$), the intercept of the odderon trajectory is smaller than 1 for most values of q_3. In particular, $\alpha_O(0) < 1$ for all real q_3. At the origin $\epsilon_3(\frac{1}{2}, q_3)$ is a continuous function of q_3 and

$$\epsilon_3(\tfrac{1}{2}, 0) = -0.738\ldots, \tag{25}$$

which indicates that the boundary conditions proposed here single out a different solution than that considered in Ref. [12]. Although their intercept, being exactly one, is bigger than that given by Eq. (25), their solution $Q(\lambda) = 1/\lambda^3$ has a strong singularity at $\lambda = 0$ and consequently would lead to a non-normalizable wave function.

At $q_3 = 0$ the Baxter equations for two and three Reggeons coincide, therefore one expects the degeneracy among the solutions. It was found in Ref. [7] that the condition of cancellation of boundary terms does not determine uniquely the two-Reggeon solution. Nevertheless, the one parameter family of solutions, for which the boundary terms cancel, contains the solution with the energy (25). Hence the correspondence between $n = 2$ and $n = 3, q_3 = 0$ sectors is maintained also at the level of solutions.

A variational estimate of the lower bound for the odderon intercept $\alpha_O(0) > 1 + 0.28 g_s^2/\pi^2$ was derived in Ref. [13]. Together with the present result it limits rather severely the allowed region of q_3 for the ground state of the system.

Because of the singularity structure seen above, the final prediction for $\alpha_O(0)$ requires, however, more detailed knowledge of the spectrum of $\hat{q}_3$. Some progress in this area has been reported in Refs. [14,15] and further work is in progress.

Our approach may be generalized to higher n [7]. Such a program would provide the leading intercept of the n Reggeized gluons as the analytic function of the $n - 1$ parameters $q_2, \ldots, q_n$.

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