Theory of Solitary Holes in Coasting Beams

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A self-consistent theory of solitary hole structures in coasting beams is presented. These phase space vortices are known from particle simulations and appear, e.g., due to a resistive wall instability. The analysis reveals new intrinsic nonlinear modes which owe their existence to a deficiency of particles trapped in the potential well, showing up as notches in the thermal range of the distribution function, where linear wave theory would predict strong Landau damping. This sheds light on the spectrum of small amplitude perturbations proving the incompleteness of linear and associated nonlinear wave theories in the kinetic regime and offers a new interpretation of recent synchrotron experiments. [S0031-9007(97)04245-2]

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Intense beams in synchrotrons that are operating close to the linear stability limits exhibit a variety of nonlinear wave phenomena. Recent experiments on debunched beams as, e.g., in the Fermilab [1,2], have unearthed a number of collective phenomena pointing to the plasma nature of the particle behavior. In fact, mode coupling and parametric effects are standard in plasma physics as are echolike phenomena. A special issue having a long tradition in plasma dynamics is the excitation of trapped particle states which appear in phase space as vortexlike structures. These states can be excited by launching an electrostatic wave leading to a process called nonlinear Landau damping [3] or appear in two-stream unstable situations where the growing waves saturate by particle trapping [4–6]. However, a linear instability is not necessarily required for the formation of these structures [7]. This suggests that there is some nonlinear wave excitation mechanism in plasma dynamics, the details of which are still to be understood.

Motivated by a similar phenomenon observed in the Fermilab experiments [2] (to be described in more detail later), we address this problem from an analytical point of view. We show that certain structures such as electron and ion holes [8] which are BGK-like (Bernstein-Greene-Kruskal) waves in plasma physics [9] can be transferred to beam dynamics as well. We shall investigate the conditions under which these states exist, analyze the resulting modes, and make a comparison with experiment.

We first calculate the self-fields acting on a beam and take for simplicity a radially uniform beam of circular cross section with radius a moving with velocity v along the axis of a circular pipe of radius b. We allow for space charge effects in the longitudinal z direction and assume that the line density λ and the fields vary as $z - v_w t$, where v_w is the relativistic phase velocity of the perturbation with $\gamma_w := (1 - v_w^2/c^2)^{-1/2} \gg 1$. Assuming a TM-mode, Maxwell's equations are readily solved in the wave frame by perturbation theory up to $0(\gamma_w^{-2})$. We get for $r \leq a$

$$B_\theta(r,z) = \frac{v_w r}{2c^2}\,\Lambda(a,z), \qquad E_r(r,z) = \frac{r}{2}\,\Lambda(a,z), \tag{1}$$

$$E_z(r,z) = E_z(z) + \left(\frac{r}{2\gamma_w}\right)^2 \partial_z \Lambda(a,z),$$

and for $r \geq a$

$$B_\theta(r,z) = \frac{v_w r}{2c^2}\,\Lambda(r,z), \qquad E_r(r,z) = \frac{r}{2}\,\Lambda(r,z), \tag{2}$$

$$E_z(r,z) = E_z(z) + \left(\frac{r}{2\gamma_w}\right)^2 \partial_z \Lambda(r,z) + \frac{q}{2\pi\epsilon_0\gamma_w^2}\,\lambda'(z)\ln\frac{r}{a},$$

where $\Lambda(r,z)$ is defined by

$$\Lambda(r,z) := \frac{q\lambda(z)}{\pi\epsilon_0 r^2} - E_z'(z), \tag{3}$$

q being the charge of a particle and ϵ_0 the vacuum permittivity. A relation between $E_z(z)$ and $E_w(z)$, the wall electric field at $r = b$, can be found by a loop integral over Faraday's law [10,11]. We obtain

$$E_z(z) = -\left(\frac{b}{2\gamma_w}\right)^2 \partial_z \Lambda\left(\frac{b}{\sqrt{g_0}}, z\right) + E_w(z), \tag{4}$$

where $g_0 := 1 + 2\ln\frac{b}{a}$. Relating $E_w(z)$ with $B_\theta(b,z)$ through the longitudinal wall impedance $\mathcal{Z}$ [12]

$$E_w(z) = -\mathcal{Z}[B_\theta(b,z) - B_\theta^0(b)], \tag{5}$$

where B_θ^0 is the magnetic field in the absence of a perturbation (i.e., $\lambda = \lambda_0, E_z = 0$), we get from (2) and (4)

$$E_z(z) + \frac{\mathcal{Z} v_w b}{2c^2}\,\Lambda_1(b,z) + \left(\frac{b}{2\gamma_w}\right)^2 \partial_z \Lambda_1\left(\frac{b}{\sqrt{g_0}}, z\right) = 0, \tag{6}$$

where $\Lambda_1(r,z)$ equals $\Lambda(r,z)$ except that λ in (3) is replaced by $\lambda_1 = \lambda - \lambda_0$, i.e., by the perturbed line density. The generalized expression (6) has two interesting

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limits. For zero impedance, as for a noninductive, perfectly conducting wall, and for long wavelength perturbations, such that $kb \ll 2\pi\gamma_w$, where k is a typical wave number, we obtain (assuming $\gamma_w \approx \gamma$ the relativistic normalized mass of a particle)

$$E_z(z) = \frac{-qg_0}{4\pi\epsilon_0\gamma^2}\,\lambda_1'(z)\,. \tag{7}$$

This well-known expression, when coupled with linearized Vlasov equation ($v_w \to \Omega/k$, where Ω is the complex frequency) yields the negative mass instability [11]. The latter assumes a beam with zero energy spread and an energy above transition energy [13]. When the limit of zero energy spread is lifted, the well-known Landau-type dispersion relation is obtained [15,16].

If, on the other hand, there is a large impedance (resistivity) Eq. (6) reduces to $\Lambda_1(b,z) = 0$ and becomes

$$E_z'(z) = \frac{q\lambda_1(z)}{\pi\epsilon_0 b^2}\,, \tag{8}$$

which is Poisson's equation.

We are interested in the limit of a large resistivity $\mathcal{Z} \approx \mathcal{R} \gg \frac{2c}{kb}$ for which linearized wave theory predicts exponentially growing waves with growth rates proportional to $\mathcal{R}$ [17]. Under such conditions [18] the excitation of a steady state structure imposed on the beam can be expected.

In the remainder we prove analytically the existence of such states, develop its characteristic properties, and show that new intrinsic modes are in play which have no counterpart in linear wave theories and their nonlinear extensions.

The beam dynamics, first of all, is governed by the Vlasov equation which reads in the frame moving with the beam [11]

$$\partial_t f + \dot{z}\partial_z f + (\dot{\Delta E})\,\frac{\partial f}{\partial \Delta E} = 0\,, \tag{9}$$

where

$$\dot{z} = R_0\Delta\omega\,, \qquad (\dot{\Delta E}) = vqE_z\,.$$

R_0 is the large radius of the design trajectory, $\omega_0 = v/R_0$ is the revolution frequency of a particle on this trajectory, and $\Delta\omega$ is the change in the revolution frequency due to a change in the particle's energy ΔE. It holds $\frac{\Delta\omega}{\omega_0} = -\frac{\eta c^2}{v^2}\frac{\Delta E}{E_0}$, where $E_0 = m_0\gamma c^2$ and $\eta = \gamma_t^{-2} - \gamma^{-2}$ being the slip factor, which is a property of the guide fields. Below transition energy, i.e., $E_0 < E_T \equiv m_0\gamma_T c^2$, η is negative.

Introducing the dimensionless quantities

$$\frac{\omega_0 t}{2\pi} \to t\,, \qquad \frac{z}{2\pi R_0} \to z\,, \qquad \frac{2\pi R_0\lambda_1}{N} \to \lambda_1\,;$$
$$u \equiv \frac{\Delta\omega}{\omega_0}\,, \qquad \varepsilon \equiv \eta q\,\frac{2\pi R_0 c^2}{v^2 E_0}\,E_z\,, \tag{10}$$

we obtain the dimensionless Vlasov-Poisson system

$$[\partial_t + u\partial_z - \varepsilon\partial_u]f(z,u,t) = 0\,, \tag{11a}$$

$$\partial_z\varepsilon = \alpha\lambda_1\,, \tag{11b}$$

with

$$\alpha := \eta\,\frac{2R_0}{\epsilon_0 E_0}\left(\frac{qc}{vb}\right)^2 N \tag{12}$$

being the space charge parameter. It carries the sign of η, is proportional to the number of beam particles N, and is independent on the sign of the charge q. The distribution function in (11a) is normalized to unity: $\int_0^1 dz\int_{-\infty}^{+\infty} du\,f(z,u,t) = 1$. If $f_0(u)$ describes the unperturbed distribution, with $\int f_0(u)\,du = 1$, it holds $\lambda_1(z,t) = \int du\,f - 1$. For an antiproton beam below transition energy, Eq. (11) corresponds to an electronic plasma system in which a positive potential hump, $\phi \geq 0$, is excited, the so-called electron hole. It holds $\varepsilon = -\partial_z\phi$. On the other hand, Eq. (11) switches to an ionic system with an excited potential well, $\phi \leq 0$, an ion hole, when the beam is above transition energy. Both situations hence admit solutions. In the following, we concentrate on the first situation [19].

Going into the wave frame, $z - \Delta ut \to z$, where Δu is the normalized phase velocity $\Delta u \equiv \frac{v_w - v}{v}$, we look for a steady state given by

$$[u\partial_z + \partial_z\phi\,\partial_u]f = 0\,, \tag{13a}$$

$$\partial_z^2\phi = -\alpha\left[\int f(z,u)\,du - 1\right]. \tag{13b}$$

The method of constructing a solution consists in two parts, in prescribing the distributions in terms of the constants of motion and in solving Poisson's equation. It has been introduced in [20] and differs from the original BGK method [9] in that instead of ϕ also the trapped particle distribution is prescribed (see below). Assuming that the unperturbed distribution is a Gaussian [21], shifted by Δu, we solve (13a) by

$$f(z,u) = \frac{1}{\sqrt{2\pi}}\begin{cases}\exp\{-\frac{1}{2}[\sigma\sqrt{u^2 - 2\phi} + \Delta u]^2\}, & u^2 \geq 2\phi > 0\,,\\ \exp(-\Delta u^2/2)\,[1 - \frac{\beta}{2}(u^2 - 2\phi)], & 0 \leq u^2 \leq 2\phi\,.\end{cases} \tag{14}$$

It is a function of the constants of motion, $\epsilon \equiv \frac{u^2}{2} - \phi$ and $\sigma \equiv \operatorname{sgn} u$ for untrapped particles (first line). The second line represents particles trapped in the potential well ($-\phi < 0$). The unperturbed state has no trapped particles and is recovered for $\phi = 0$. Note that β controls the state of trapped particles, depletion zones in

f being represented by negative β's [22]. Integration of f over u yields the line density [19]

$$\lambda = 1 - \frac{1}{2} Z_r'\left(\frac{\Delta u}{\sqrt{2}}\right)\phi - \frac{4}{3} b(\beta, \Delta u)\phi^{3/2} + \ldots, \tag{15}$$

valid for small ϕ. Z_r is the real part of the plasma dispersion function and b stands for $b(\beta, \Delta u) := \frac{1}{\sqrt{\pi}}(1 - \beta - \Delta u^2)\exp(-\frac{\Delta u^2}{2})$. Note that β should not be mixed up with v/c. Expressing the right-hand side of (13b) by $-V'(\phi)$ we can integrate (13b) once and obtain the "energy law" $\phi'^2/2 + V(\phi) = 0$ with $V(\phi)$ given by $V(\phi) = |\alpha|[+\frac{1}{4}Z_r'\phi^2 + \frac{8}{15}b\phi^{5/2}]$. We assume $\alpha = -|\alpha| < 0$ and $V(0) = 0$. V has to be negative in the interval $0 < \phi < \psi$ and has to vanish at $\phi = \psi$, the maximum excursion of the bell-shaped potential structure. The latter condition yields

$$-\frac{1}{2} Z_r'\left(\frac{\Delta u}{\sqrt{2}}\right) = \frac{16}{15} b\sqrt{\psi}, \tag{16}$$

which is the *nonlinear dispersion relation* as it yields Δu in terms of β and ψ and hence b. With the help of (16) $V(\phi)$ becomes

$$-V(\phi) = \frac{8|\alpha|b}{15}\phi^2\left(\sqrt{\psi} - \sqrt{\phi}\right). \tag{17}$$

$V \leq 0$ implies $b > 0$. From what follows that for moderate $\Delta u \leq 1$, β has to be negative. Hence the distribution function must be depleted in the trapped particle range. From the energy law we obtain by a quadrature

$$\phi(z) = \psi \operatorname{sech}^4 \sqrt{\frac{|\alpha|b\sqrt{\psi}}{15}}\, z. \tag{18}$$

Equations (16)–(18) represent the solution to our problem. A numerical example is given by $\psi = \frac{1}{400}, \beta = -31, \Delta u = 0.2$, noting that this triplet solves (16). The corresponding distribution (14) is plotted in Fig. 1, showing a definite notch at $u = \Delta u$ within the trapped particle range, i.e., in the interval between the dotted lines. Notice that such a solution should not exist from a linear point of view due to strong Landau damping in this velocity range. In phase space the solution (14) is vortexlike with a depleted zone at resonance. Figure 1 also shows the deviation of f from Gaussian which has a *finite* velocity gradient (see below).

High intense beams in synchrotrons near their stability limits represent an "ideal test bed for investigating these coherent phenomena" [1,2]. In order to assess the possible cause of fluctuations that are ubiquitous in any of such beams, the authors of Ref. [2] performed beam transfer function measurements in the Fermilab Main Ring and found a decidedly nonlinear beam response. Most remarkable was the observation of "sharp gaps or notches in the response function which presumably correspond to depletion zones in the momentum distribution function."

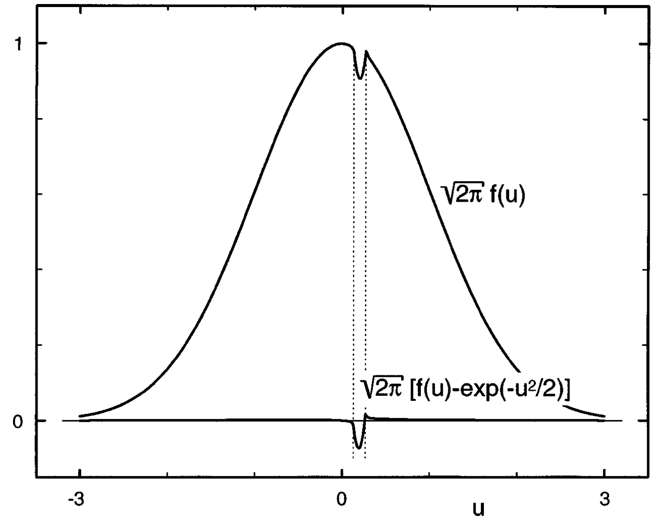

FIG. 1. The distribution $f(u)$ and its deviation from the unperturbed state as functions of u at $\phi = \psi$.

Although transverse resonances may have determined the location of notches and may have contributed to their depletion in momentum space, as suggested by the authors, there are good reasons to believe that, at least in the saturated state, the observed structures are holes as described above. This conclusion is supported by the following observations: (i) The excitation of sidebands downward in frequency with an enhancement, independent of the drive frequency and whether the machine was above or below transition energy, near the notches; (ii) the lack of any harmonic coupling; (iii) the measurement of characteristic beam current oscillations, after the drive was switched off, reminiscent to current oscillations in particle simulations involving wave overturning and particle trapping; and (iv) this behavior was observed at the lowest measurable signal levels.

The last observation is especially interesting because it can be interpreted as experimental proof of the following theoretical affirmation: independent of their strength, the present wave structures have no connection to linear wave solutions or their nonlinear descendants. This is easily seen from (13a) or (14). At resonance, $u = 0$, we have $\partial_u f = 0$. Splitting f into $f_0 + f_1$, we readily see that $|\partial_u f_1| = |\partial_u f_0|$. This means that a linearization procedure, which consists in neglecting the $\partial_u f_1$ term in the Vlasov equation, is not justified, no matter how small the amplitude, ψ, is. Even at extremely small amplitudes, nonlinearity will prevail. In addition, as the extension to periodic waves with finite wavelength [19] shows, the solution remains *nonlinear* even in the harmonic limit. In the kinetic regime, the presence of resonant particles destroys the equivalence between linearity and harmonicity. Mathematically speaking, the two procedures, A := take the small amplitude limit and B := solve the equation of motion, do not commute.

Linear wave theories which in succession use AB (A first then B) differ from the present handling, BA, of the equations. They are incomplete as the present modes and their periodic generalizations are missed in their spectrum. In particular, linear wave theories, such as van Kampen or Landau theories as well as their nonlinear extensions, are inadequate the moment resonant particles are involved. This holds even for *arbitrarily small* amplitudes. This important and surprising fact does not yet appear to have been recognized in the literature. For more details, e.g., its consequences for transport, see [19].

We finally mention that (18) with z replaced by $z - \Delta ut$ is a steady state solution of the following evolution equation [23]:

$$\phi_t + \left(c_0 + c_1\sqrt{\phi}\right)\phi_z + \phi_{zzz} = 0\,, \qquad (19)$$

with $c_0 = \Delta u - \frac{16}{15}b|\alpha|\sqrt{\psi}$ and $c_1 = 2|\alpha|b$. This modified Korteweg–de Vries equation has some signatures of integrability [24] which could justify the notation "soliton." However, as known from simulations, holes or vortices in phase space tend to coalesce and lose their identity (when they approach each other at small speeds). Hence, we prefer the notation "holes."

A central issue in follow-on works will be the stability of these equilibria to establish its longevity or perhaps to find recipes for construction families of nonlinear structures that span a wide range of "dissipation" time scales. Numerically, these solitary structures were found to be stable in 1D [19,25], showing also some robustness in 2D [26]. Analytically, however, the problem of stability is still unsolved. It remains one of the major theoretical challenges. One reason is the nonlocal character and the lack of self-adjointness of the spectral operator obtained within a linear stability analysis [8,27].

In summary, the existence of a new type of solitary structures in coasting beams has been shown. These solutions owe their existence to a deficiency of particles trapped in the self-generated potential trough. They lie outside the realm of linear wave theories and their nonlinear descendants, and have, therefore, no relation to linear waves such as slow or fast space charge waves.

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[22] The critical reader will realize that f in Eq. (14) is continuous but nonanalytic in phase space. In fact, its first derivatives diverge at the separatrix, given by $\epsilon = 0$. However, f can be viewed as a representative of an analytic smooth function $\hat{f}$ with the same macroscopic properties. Following Ref. [19], $\hat{f}$ is found, e.g., by a Fourier-Hermite expansion of f and by a subsequent truncation of the series. That $\hat{f}$ is indeed an exact solution of the Vlasov-Poisson system can be seen from Fig. 2(c) of [19].

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