Self-Similar Energy Decay in Magnetohydrodynamic Turbulence

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The self-similar decay of energy in magnetohydrodynamic flows is examined in the spirit of the Kolmogorov phenomenology while incorporating in the expression for the energy transfer to small scales the interactions between turbulent eddies and Alfvén waves due to a large-scale magnetic field. The model is parameter-free and does not rely on the existence of the several invariants of ideal MHD except energy. High-resolution numerical simulations in two dimensions confirm its validity in a way that leads as expected to a decay significantly slower than for neutral fluids. [S0031-9007(97)04201-4]

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Many predictions due to Kolmogorov [1] for homogeneous and isotropic turbulent flows have been verified for laboratory, atmospheric, and numerical data. In this Letter we modify one such law, namely that concerning the self-similar decay of energy in dealing with conducting flows at high kinetic and magnetic Reynolds numbers, as relevant in many instances in geophysics and astrophysics where magnetic fields are dynamically important.

In the fluid case, assuming that the kinetic energy of the flow E^V—after the initial onset, and before the final period of decay—decreases in time in a self-similar manner, one can deduce [1] that $E^V(t) \sim (t - t_*)^{-10/7}$, where t_* is typically the time at which the enstrophy $\langle \boldsymbol{\omega}^2 \rangle$ reaches its first maximum, with $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ the vorticity. The derivation relies on the invariance of the Loitsianskii integral $\mathcal{L} \sim v_{\mathrm{rms}}^2 \int_0^\infty r^4 f(r)$, where $f(r)$ is the longitudinal correlation function of the velocity. Let us give here a simple version of the argument. At low wave number, the kinetic energy spectrum $E^V(k) \sim k^s$ (with $s = D + 1$, where D is the space dimension) up to $k_0 = 2\pi/\ell$ after which the Kolmogorov law $E(k) \sim k^{-5/3}$ begins; ℓ is identified with the integral scale of the flow [2]. This large-scale spectrum is linked to small-scale beating or backscattering and, with sufficient scale separation, dominates the total energy which can thus be evaluated as $E^V = \rho_0 \langle v^2 \rangle /2 \sim \ell^{-(s+1)}$ with ρ_0 the constant density. Take now $\langle v^2 \rangle \sim (t - t_*)^{-\alpha_s}$ and $\ell \sim (t - t_*)^{\beta_s}$; thus, $\alpha_s = \beta_s(s + 1)$ since $v^2 \ell^{s+1}$ is assumed constant during this temporal phase. Differentiating $E^V(t)$ leads to $1 - \beta_s = \alpha_s/2$, where the Kolmogorov relationship $\epsilon \sim v^3/\ell$ has been used with $\epsilon = -\dot{E}^V$ the kinetic energy transfer rate to small scales. The decay rates thus obtain with $(s + 3)\alpha_s = 2(s + 1)$ and $(s + 3)\beta_s = 2$. In dimension three ($s = 4$), the Kolmogorov law $E(t) \sim (t - t_*)^{-10/7}$ follows, a law backed up by experiments [3] and by closure computations for turbulent flows [4,5].

At least two possibilities arise, in incompressible MHD, as how to construct a phenomenology for decaying conducting flows that obey the equations

$$(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P_* + \nu \nabla^2 \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{b}\,, \quad (1)$$

$$(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{b}\,, \quad (2)$$

with $\mathbf{b}$ the magnetic induction, P_* the total pressure, ν the viscosity, η the magnetic diffusivity, and $\nabla \cdot \mathbf{v} = 0$, $\nabla \cdot \mathbf{b} = 0$. On the one hand, one may take into account the presence of more than one invariant of the equations in the nondissipative case leading to a modification of the large-scale energy spectrum because of inverse transfer—of the squared magnetic potential $E^A = \langle A^2 \rangle$ for $D = 2$ (where $\mathbf{b} = \nabla \times \mathbf{A}$), and for $D = 3$ of magnetic helicity $H^M = \langle \mathbf{A} \cdot \mathbf{b} \rangle$ [6–9]. The point of view adopted here is rather to ignore this inverse transfer to the large scales, under the assumption that it does not act directly on the direct cascade of energy, i.e., the (classical) assumption of independence of inertial ranges for large-scale separation between the maximum scale of the flow, the integral and dissipative scales. The phenomenology is thus constructed in the spirit of K41 but incorporating the specificity of nonlinear interactions in MHD with weak velocity-magnetic field correlations as proposed by Iroshnikov and by Kraichnan [10]. A somewhat similar approach can be found in [8] where α_s and β_s are left open and the large-scale spectrum dependency $E(k) \sim k^s$ is not considered, but rather studying the effects of several parameters such as the presence of a uniform magnetic field $\mathbf{B}_0$, and a nonzero correlation coefficient ρ_C between the velocity and the magnetic field with $\rho_C = 2\langle \mathbf{v} \cdot \mathbf{b} \rangle / \langle v^2 + b^2 \rangle$. No such explicit dependency on either $\mathbf{B}_0$ or ρ_C is studied here (it will be reported in a forthcoming paper [11]), the focus being on developing the simplest version of decay laws that is compatible with well-resolved numerical data.

Modifying the Kolmogorov analysis in order to take into account the slowing down of the energy transfer due to the interaction with the dynamically self-consistent large-scale magnetic field of amplitude b_0 leads [10] to an energy spectrum $E^{\pm}(k) \sim (\epsilon_B b_0)^{1/2} k^{-3/2}$, where $E^{\pm}$ is the energy of the Elsässer field $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$. The new temporal decay laws [hereafter referred to as the IK (Iroshnikov-Kraichnan) model] obtain straightforwardly [12]. We assume as before that in the large scales

$$E^{\pm}(k) \sim k^s, \quad (3)$$

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and that the $\pm$ energies and integral scales follow self-similar laws (with $z^+ \sim z^- \sim z$ in the uncorrelated case)

$$z^2 \sim (t - t_*)^{-\alpha_{B,s}}, \qquad \ell \sim (t - t_*)^{\beta_{B,s}}. \tag{4}$$

In the IK phenomenology, one replaces the K41 expression for energy transfer by

$$\epsilon_B = -z^2/\tau_{tr} = -z^4/\ell b_0 , \tag{5}$$

where the transfer time τ_{tr} is evaluated with a simple rule, namely, $\tau_{tr} = \tau_{NL}(\tau_{NL}/\tau_A)$ with $\tau_{NL} = \ell/z_\ell$ and $\tau_A = \ell/b_0$, respectively, the turn over and Alfvén times [13]. This choice differs from what is done in [8]: Keeping the transfer time based on the hydrodynamical time, together with following the von Kármán-Howarth [14] approach, one then finds a $(t - t_*)^{-1}$ decay law both for fluids and MHD.

From (5) we obtain $-\beta_{B,s} + 1 = \alpha_{B,s}$. Using the constancy of $z^2\ell^{s+1}$, this leads to the new scaling exponents

$$\alpha_{B,s} = \frac{s+1}{s+2}, \qquad \beta_{B,s} = \frac{1}{s+2}. \tag{6}$$

Equations (6) differ substantially from their Kolmogorov analogs in a way compatible with the IK phenomenology: Energy decays less efficiently in MHD. No experimental data are available at high magnetic Reynolds number, so that numerical simulations are particularly useful in differentiating between the IK and K41 behaviors, possibly using some sort of modelization such as in sparse methods with a reduced set of wave numbers [15]. Such a differentiation may also be amenable to verification using shell models of turbulence as introduced in [16], but modified for MHD. Finally note that taking into account the correlation between the velocity and the magnetic field [17], one must *a priori* distinguish between the temporal evolution of the $E^\pm$ spectra with transfer rates $\epsilon_B^\pm = -\dot{E}^\pm$. However, when extending the IK theory to the correlated case one finds $\epsilon_B^+ = \epsilon_B^- = \epsilon_B$. It is then straightforward to show [18] that the decay laws of $E^\pm$ are independent of correlation.

Numerical simulations in three dimensions using periodic boundary conditions [8,9] indicate that $\alpha_{B,4} \sim 0.95$, to be compared with our theoretical prediction of $\frac{5}{6}$; such computations should be run for longer times and at higher Reynolds numbers, for example implementing symmetries as in [19]; indeed, one recalls that at a low Reynolds number, the decay law is blurred by linear dissipation and thus appears steeper, as also seen in the computations presented in Fig. 1. In 2D as well [6,20,21], slowing down of transfer to small scales is observed. A check of the data of the computations reported in [20] yields a similar slower temporal dependence using either deterministic initial conditions with $\rho_C = 50\%$ or random fields with $\rho_C \sim 10\%$.

In order to improve the assessment of the model, we report here on a new series of two-dimensional numerical simulations using a hyperviscosity algorithm, a standard procedure whereby the Laplacian in the dissipative terms

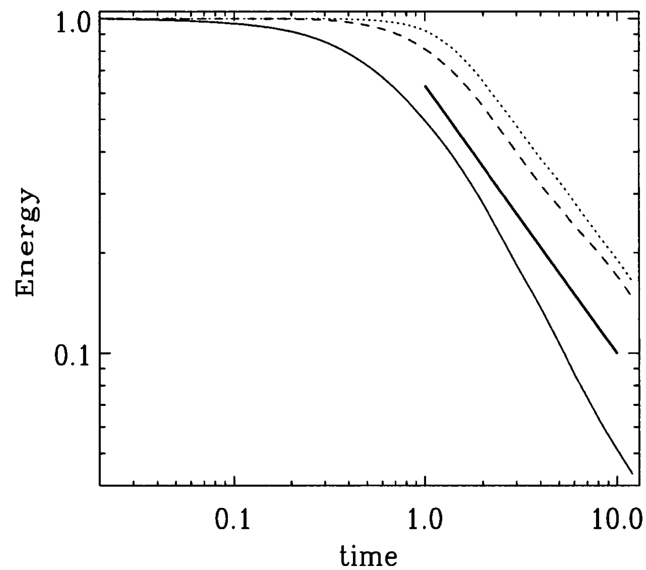

FIG. 1. Temporal decay of E^+ (normalized) for $\gamma = 1$ (solid line) for the primitive MHD equations, and $\gamma = 2$ (dashed line) and $\gamma = 8$ (dotted line). The heavy solid line corresponds to (6) with $s = 3$.

in the MHD equations is replaced by $(-1)^{\gamma+1}\lambda_\gamma\nabla^{2\gamma}$, with the same γ-dependent dissipation coefficient λ_γ in the velocity and magnetic induction equations (effective unit magnetic Prandtl number), and with $\gamma \geq 1$. This form of tempering with small scales when $\gamma \neq 1$ has been tested in MHD [22] and may in fact be justified in the light of the exact results obtained recently [23] in the framework of the passive scalar—with Gaussian white noise in time velocity field and forcing—whereby it can be shown that, at the statistical level, averaging does produce an eddy diffusivity that swamps the molecular or numerical one, as expected on physical grounds.

The runs use a standard pseudospectral code, the correlation between the velocity and magnetic field being small throughout ($\rho_C \sim 0.05$). Initially, the fields have random phases, kinetic and magnetic energies are comparable with $E^T = 1$, and the spectra are given by $E^\pm(k) \sim k^s \exp[-(k/k_0)^2]$ with $s = 3$ and with a maximum at $k_M \sim 10$ for E^+ and $k_M \sim 11$ for E^-. The dissipation coefficients are, on the grid of 512^2 points, $\lambda_1 = 2 \times 10^{-3}$, $\lambda_2 = 1 \times 10^{-7}$, and $\lambda_8 = 1 \times 10^{-34}$; higher resolution runs to be reported in [11] for $\gamma = 1$ confirm these results with a power-law fit that progressively approaches the predicted law (6) as λ_1 varies by up to a factor of 3.

Temporal decay of E^+ is given in Fig. 1 in log-log coordinates with $\gamma = 1$ (solid line), $\gamma = 2$ (dashed line), and $\gamma = 8$ (dotted line); the thick solid line has the theoretical slope of $\alpha_{B,3} = \frac{4}{5}$. For the same runs, the temporal evolution of the total squared magnetic potential E^A is such that at $t = 20t_*$, it has decayed by 51% for $\gamma = 1$, by 2% for $\gamma = 2$, and by 0.3% for $\gamma = 8$. The effect of hyperviscosity is thus quite clear: The onset of decay is delayed as γ grows, to the time $t_* \sim 0.9$ of maximum total enstrophy $\langle \boldsymbol{\omega}^2 + \mathbf{j}^2 \rangle$, where $\mathbf{j} = \nabla \times \mathbf{b}$ is the current density.

A least-squares fit for the temporal evolution of the total energy yields $\alpha_{B,3} = 1.06 \pm 0.02$ for $\gamma = 1$, $\alpha_{B,3} = 0.70 \pm 0.01$ for $\gamma = 2$, and 0.77 ± 0.01 for $\gamma = 8$, in striking contrast with the K41 value. For the growth of the integral scale, one finds $\beta_{B,3} = 0.42 \pm 0.01$ for $\gamma = 1$, $\beta_{B,3} = 0.34 \pm 0.02$ for $\gamma = 2$, and 0.32 ± 0.02 for $\gamma = 8$, whereas the theoretical value is $\frac{1}{5}$. Analysis of the data in terms of the Elsässer variables gives similar results. The lesser agreement for the growth of the integral scale may be linked to the presence of an inverse transfer of magnetic potential, felt insofar as scale separation is insufficient and thus the $E(k) \sim k^3$ law is valid for an insufficient span of wave numbers. Note that the difference between the K41 and IK phenomenologies is in the relationship between the two coefficients α and β: In the former case $1 - \beta_s = \alpha_s/2$, whereas in the latter $1 - \beta_{B,s} = \alpha_{B,s}$. The latter relationship is indeed better verified by the data presented here.

In three dimensions, the model predicts an energy decay which follows at $(t - t_*)^{-5/6}$ law as opposed to $(t - t_*)^{-10/7}$ for K41. In [8] on the other hand, the authors conclude that, *both for fluids and MHD*, $E(t) \simeq (t - t_*)^{-1}$, a result backed up by low-resolution simulations which should be pursued at higher resolution to sort out fluid (Kolmogorov-like) and MHD effects.

One can also consider higher moments of the velocity and magnetic fields, for which decay laws can be derived as well. For example, it is readily shown that for the present IK model, the generalized palinstrophies $\mathcal{P}^{\pm} = \langle \nabla \times (\nabla \times \mathbf{z}^{\pm}) \rangle$ decay as $\sim (t - t_*)^{p_s}$ with $(s + 2)p_s = s + 5$. Similarly, one finds

$$\langle |\boldsymbol{\omega} \pm \mathbf{j}|^q \rangle \sim (t - t_*)^{-\delta_{q,s}}, \tag{7}$$

with $\delta_{q,s} = q \ \forall \ s$ in 2D neutral fluids ($\mathbf{j} = 0$), following the Batchelor phenomenology [24], and

$$\delta_{q,s} = \frac{(s + 3)}{2(s + 2)} q \tag{8}$$

for MHD in the IK case [25]. For the run with $\gamma = 8$, we find $p_3 = 1.20 \pm 0.03$ compared to the theoretical prediction of $\frac{8}{5}$; similarly, one has $\delta_{2,3} = \frac{6}{5}$ for $q = 2$, to be compared with $\delta_{2,3} = 1.09 \pm 0.02$ for the present computations (and in [21], of 1.30 ± 0.04). Intermittency effects, as modeled, e.g., in [26] for MHD, may have to be taken into account when considering higher order moments.

Several extensions of the present model can be envisaged: (i) Express the nonuniversality of the large-scale spectrum (see [2]) by letting s as a free parameter; (ii) model the case of constant integral scale for neutral fluids as in [5] (this can be easily implemented letting $s \to \infty$); and finally, (iii) mimic slower transfer rates to small scales, stemming, e.g., from growth of v-b correlations [18] (see [13] as well in a different context). The initial ratio of kinetic to magnetic energy may be relevant as well as it is known to lead to different regimes [27]. These issues require a large parametric study and will be tackled in the future [11]. The present computations, at least in 2D, seem sufficient to rule out the -1 law predicted in [6] with the assumption of constancy of $\langle \mathbf{A}^2 \rangle$, an assumption well fulfilled at high hyperviscosity with however a $\sim (t - t_*)^{-0.77}$ decay law, a decay almost coincident with that reported in [21] under the same conditions of constancy of $\langle \mathbf{A}^2 \rangle$.

The lack of dependence in (6) on the global velocity-magnetic field correlation coefficient ρ_C is not necessarily paradoxical if one recalls that the correlation is not positive definite: Even a moderate global value of ρ_C can in fact hide high local values of opposite signs. It is known that correlations develop in time [17], and that the flow organizes in several bipolar set of regions with strong $\pm$ values for ρ_C, i.e., with correspondingly weak nonlinearities except at the border of such regions where all small-scale activity takes place, in particular, in the formation and disruption of current and vorticity sheets [28]. In other words, nonlinearities in MHD are self-defeating, and thus MHD may turn out to be simpler than its fluid counterpart: A state of weak nonlinearities seems to be attractive, in apparent contrast to Beltrami ($\mathbf{u} = \pm \boldsymbol{\omega}$) flows. On the other hand, such conclusions were drawn on the basis of closure models of turbulence and of numerical simulations still at moderate Reynolds numbers. Transitions at a high Reynolds number towards other types of attractors cannot be ruled out [29]; in that light, numerical experiments using hyperviscosities are useful in exploring in a modelized way this high Reynolds number range. Similarly, the cases of both low and high magnetic Prandtl numbers should also be considered.

Finally, the inclusion of a uniform magnetic field $\mathbf{B}_0$ has profound effects on the dynamics, as clearly observed in [8], leading, e.g., to anisotropies [30]. It is also claimed that it suppresses small-scale turbulence, including when it is well below equipartition with the velocity [13], and thus strongly alters transport properties. Such claims are done in the framework of the dynamo problem, i.e., with a forcing term included in the MHD equations in order to observe growth of the magnetic field, a setting which *stricto sensu* does not apply here. Note, however, that neither the Kolmogorov nor the IK phenomenologies makes explicit use of transport coefficients, but simply uses dimensional constraints, so that this may not directly affect the energy decay law, once the proper dimensional formulation for the energy transfer rate as in (5) is given.

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