Colliding Waves in a Model Excitable Medium: Preservation, Annihilation, and Bifurcation

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We analyze the transition from annihilation to preservation of colliding waves. The analysis exploits the similarity between the local and global phase portraits of the system. The transition is shown to be the infinite-dimensional analog of the creation and annihilation of limit cycles in the plane via a homoclinic Andronov bifurcation, and has parallels to the nucleation theory of first-order phase transitions. [S0031-9007(97)04203-8]

Waves in physical, chemical, and biological systems are broadly classified into two categories: classical and excitable. Some distinguishing qualitative features that characterize excitable waves include their mutual annihilation following a collision and their inability to reflect from boundaries [1]. However, recent experimental and numerical work in different excitable systems [2] show the existence of a transition between the annihilation and preservation of pairwise colliding waves as a function of the governing parameters in the problem. In this Letter, we examine this transition for a model problem and show that it is an infinite-dimensional analog of a homoclinic Andronov bifurcation [3]. The geometric arguments used imply that this transition is generic to many other excitable systems.

To illustrate the main ideas, we focus on a simple dynamical system associated with a continuum description of a chain of pendula coupled by torsional springs, subjected to a constant scaled torque Ω and a dimensionless viscous damping ν. The equation of motion for this system is given by

$$\theta_{tt} + \nu\theta_t + \sin\theta = \Omega + \theta_{xx}. \tag{1}$$

Here the scaling assumes that the inertial and coupling terms appear at leading order. When $\Omega = \nu = 0$, (1) reduces to the sine-Gordon equation which is completely integrable and exhibits solitons that interact elastically. In the overdamped limit, the inertial term is dominated by the viscous damping term. Then (1) reduces to the reaction-diffusion equation

$$\theta_t + \sin\theta = \Omega + \theta_{xx}, \tag{2}$$

which is a prototype of a dynamical system which exhibits excitable waves [4]. If one considers the variable $\sin(\theta)$ instead of θ itself, these waves are pulses, not fronts; we will call them fronts from now on. We carried out a number of numerical experiments on (1) to understand this transi-

tion from dispersive to diffusive behavior. The numerical method consisted of a central-difference scheme in space and a Runge-Kutta method in time with either periodic or Neumann (zero flux) boundary conditions. The size of the integration domain was large compared to the width of the fronts, which is proportional to $\nu^{-1/2}$. The initial conditions corresponded to two fronts that are far apart and moving towards each other. For ν large enough, the fronts collide and annihilate each other. For $\nu \ll 1$, the fronts go through each other following the collision. In this limit, which corresponds to the nearly integrable case, our computations agree with analytical and numerical results for the perturbed sine-Gordon equation [5]; in the excitable case considered here, we are far from integrability. In the case with periodic boundary conditions the fronts collide twice every period. When Neumann boundary conditions are used, the fronts are reflected at the boundaries and collide periodically in the center of the domain. For fixed Ω, as $\nu \sim \nu_c$, the critical value of damping where this transition between annihilation and preservation occurs, the two fronts coalesce into a near stationary solution. As $\nu \rightarrow \nu_c$, the time spent close to this limiting solution diverges. This points to the importance of understanding the stationary solution of (1) and its temporal stability.

Before going further, we first consider the simpler dynamical system corresponding to the spatially homogeneous counterpart of (1), which shares many of the properties of the extended system. Dropping the coupling term in (1) yields the equation of motion for a damped forced pendulum

$$\theta_{tt} + \nu\theta_t + \sin\theta = \Omega. \tag{3}$$

For large ν, the inertial terms are unimportant and the only stable solution (modulo 2π) is $\theta_s = \sin^{-1}\Omega$; the other fixed point $\theta_u = \pi - \sin^{-1}\Omega$ is unstable. As Ω is increased, the stable (node) and the unstable (saddle) fixed points coalesce through a saddle-node bifurcation [3]

when $\Omega = 1$. Close to this bifurcation, the period diverges algebraically as $O((1 - \Omega)^{-1})$. For $\Omega > 1$ and arbitrary ν, the only solution is a (rotating) periodic solution (limit cycle). For $\Omega \ll 1$ and for sufficiently large ν, the only stable solution is $\theta = \sin^{-1} \Omega$. This global attractor (modulo 2π) is excitable, since a finite excitation can lead to a large excursion in the phase space corresponding to a rotation. However, as ν is decreased below a critical value $|\nu_o(\Omega)|$, both the periodic solution and the stationary solution can coexist. This can be understood in terms of the potential $V(\theta) = -\cos\theta - \Omega\theta$ associated with (3). In the presence of large damping, all initial conditions lead to trajectories that end up at the bottom of a valley. For the critical value of $\nu = \nu_o(\Omega)$, the homoclinic orbit connecting the top of nearby hills occurs giving birth for a slightly smaller value of ν to a limit cycle with a large period [3] which diverges logarithmically in the variable $\nu - \nu_o$. The formation of the homoclinic orbit corresponds to the transition to hysteretic behavior (3), which as we will see corresponds to the region in parameter space when colliding front solutions to (1) are preserved. In Fig. 1 we show the "phase" diagram that illustrates the various states of (3) along with their stability boundaries. It is interesting to interpret this phase diagram in the context of nucleation phase transitions [6]. The order parameter of the transition to rotation is defined as the frequency of the rotating pendulum. In Fig. 1, the vertical line $\Omega = 1$ above the point I corresponds to a (continuous) second-order transition; along it one has a saddle-node bifurcation,
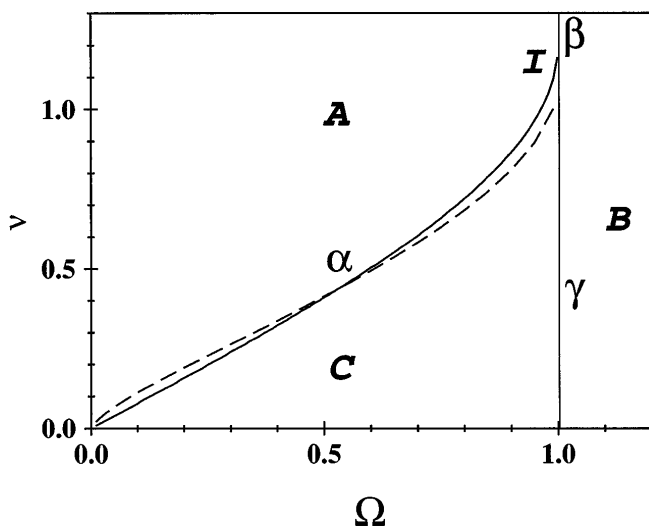

FIG. 1. Phase diagrams obtained numerically showing the existence of various solutions for (1) and (3). The line represents $\nu_o(\Omega)$ computed by a shooting method for the homogeneous equation (3), and the dashed line represents $\nu_c(\Omega)$ obtained from numerical simulations of the extended system (1). A: Stable stationary solution. B: Stable periodic solution. C: Both solutions are metastable. α: Homoclinic bifurcation. β: Saddle-node bifurcation (second-order phase transition). γ: Saddle-node bifurcation (first-order phase transition). I: Codimension-2 Andronov bifurcation point (weakly first-order phase transition).

and the period of rotation diverges algebraically. The vertical line $\Omega = 1$ below point I corresponds to a first-order transition since the bifurcation leads to a limit cycle with a finite period. The point I is a codimension-2 point where two Andronov bifurcations occur simultaneously and is a tricritical point in the jargon of phase transitions. A similar scenario occurs in the transition to chaos through intermittency [7] which changes from second order to first order [8] in a two-parameter family of dynamical systems.

Returning to the extended system (1), we consider the stationary solution for $\nu = \nu_c$ that characterizes the transition from annihilation to preservation. We will interpret this stationary solution as a nucleation bubble [9] familiar in the theory of first-order phase transitions. To compute this solution we write the time-independent version of (1) as

$$\theta_{xx} + \frac{dW}{d\theta} = 0,\qquad (4)$$

where $W = -V = \Omega\theta + \cos\theta$ represents a fictitious tilted potential. The separatrix solution of (4) with boundary conditions, $\theta \to \sin^{-1}\Omega$, $\theta_x \to 0$ as $x \pm \infty$, corresponds to the nucleation bubble, and must be computed numerically in general. However, it can be computed analytically in the two limits, $\Omega \to 0$ and $\Omega \to 1$. We focus on the case $\Omega \sim 1$ when (1) has excitable properties. Substituting $\Omega = 1 - \epsilon$ along with the expansion $\theta(x) = \frac{\pi}{2} + \epsilon^{1/2}\Theta^{(1)}(\epsilon^{1/4}x) + \dots$ in (4), at $O(\epsilon)$ we get $\Theta^{(1)}_{XX} = 1 - \Theta^{(1)2}/2$, where $X = \epsilon^{1/4}x$. The separatrix solution of this equation, which we denote by $\Theta^{(1)}_N(X)$, can be expressed in terms of cnoidal elliptic functions [10]. To study the stability of this solution we substitute $\theta(x,t) = \frac{\pi}{2} + \epsilon^{1/2}\Theta^{(1)}_N(\epsilon^{1/4}x) + \cdots + \epsilon^{1/2}\Psi(\epsilon^{1/4}x, \epsilon^{1/2}t)$ in (1) and linearize the resulting equation about $\Psi = 0$. At leading order, we find that the perturbation $\Psi(\epsilon^{1/4}x, \epsilon^{1/2}t)$ satisfies the following Schrödinger equation:

$$\nu\Psi_T = -\Theta^{(1)}_N(X)\Psi + \Psi_{XX},\qquad (5)$$

with $T = \epsilon^{1/2}t$. Since the potential associated with $\Theta^{(1)}_N$ is compact, it is sufficient to study the discrete spectrum of the Schrödinger operator with a potential $\Theta^{(1)}_N$ to determine the stability of the nucleation solution $\Theta^{(1)}_N$. Because (1) is invariant with respect to spatial translation, the linearized problem (5) has a zero eigenvalue. The corresponding eigenfunction is $\partial_X \Theta^{(1)}_N$ and has a single node. Therefore the fundamental eigenfunction with no nodes corresponds to the only positive eigenvalue by the ordering lemma for the Schrödinger operator [11]. Thus the stable manifold W_s of the nucleation solution has codimension 1.

In the functional "phase portrait" sketched in Figs. 2(a)–2(c) for the case with periodic boundary conditions, the nucleation solution corresponds to the saddle. The variation
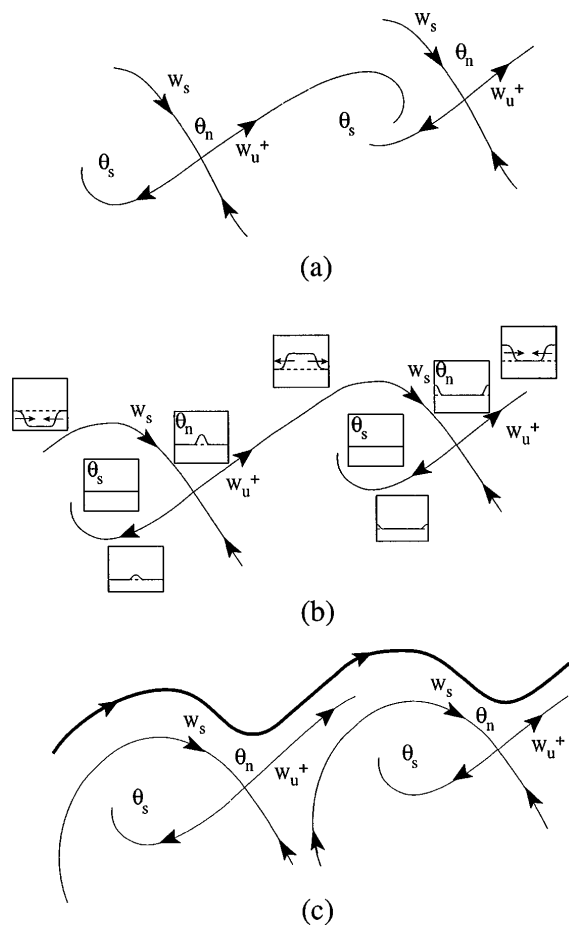
FIG. 2. Projection of the functional phase portrait of (1). θ_n is the nucleation solution and θ_s is the homogeneous stable stationary solution. (a) Before the homoclinic bifurcation, $\nu > \nu_c$, the two fronts annihilate each other. (b) At the bifurcation point, $\nu = \nu_c$, the pair forms a stationary solution. Evolution of the pair of fronts along the trajectories is also presented. (c) Once the threshold is crossed, $\nu < \nu_c$, the fronts go through each other.

in the damping term ν leads to the transition from preservation to annihilation of colliding fronts. The codimension 1 stable manifold W_s acts as a separatrix and locally separates the infinite-dimensional phase space of the flow associated with (1) into two distinct regions corresponding to initial conditions that lead to either the propagation of two fronts or the decay of an excitation. This qualitative geometrical picture is quite independent of the particular model considered but must be modified for systems that are not periodic in the dependent variable θ. In (1), however, due to this periodicity, the entire collision process occurs on the manifold W_u^+ which describes the propagation of two fronts initiated from the nucleation solution θ_n. Then the transition from annihilation to preservation reduces to the relative position of W_u^+ with respect to the stable manifold W_s. In Fig. 2(a), we have depicted the case $\nu > \nu_c$, where the collision leads to the annihilation of the fronts. In this case W_u^+ converges to the homogeneous stable excitable state θ_s. Figure 2(b) corresponds

to the critical situation $\nu = \nu_c$ where W_u^+ converges to the nucleation solution. Since W_u has a codimension one, this occurs generically for a one-parameter family of dynamical systems. This homoclinic Andronov bifurcation of the nucleation solution is the infinite-dimensional analog of the planar homoclinic bifurcation for (3) and is responsible for the preservation-annihilation transition of the colliding fronts. In Fig. 2(c), the phase portrait corresponds to the case $\nu < \nu_c$ when the fronts are preserved following the collision (modulo 2π). The corresponding limit cycle describes the periodic process of the successive collisions of the two fronts in a periodic geometry.

In order to make this qualitative picture more quantitative, we have computed the "residence" time T that the colliding fronts spend close to the nucleation solution, for $\nu \sim \nu_c$. In Fig. 3(a), we plot T versus ν when $\Omega \sim 1$, and observe the similarity to the λ point in second-order phase transitions, in accordance with the switch between first- and second-order transitions close to $\Omega \sim 1$, $\nu \sim \nu_c$, shown in Fig. 1. In Fig. 3(b), we plot T versus $\ln|\nu_c(\Omega) - \nu|$. As expected in the case of a homoclinic bifurcation, we see that $T \sim -\frac{1}{\sigma_1} \ln|\nu_c(\Omega) - \nu|$, with $\sigma_1 = 0.249 \pm 0.004$. For a typical value of $\Omega = 0.97$ ($\nu_c = 0.939$), corresponding to this numerical experiment we have also numerically determined the spectrum of the nucleation solution. The leading eigenvalues are $\sigma_1 = 0.250\,484$, $\sigma_0 = 0.000\,045$, and $\sigma_{-1} = -0.279\,570$; we observe that σ_0 is the eigenvalue of the translational mode to within numerical error, and σ_1 is in agreement with the slope of the semilog plot in Fig. 3(b). Since $|\sigma_1| \ll |\sigma_{-1}|$, this computation confirms the observed stability of the limit cycle [12] for $\nu \lesssim \nu_c$.

A convenient way of visualizing solutions to (1) is to plot the potential surface $V(\theta, x) = -\Omega\theta - \cos\theta$ and to
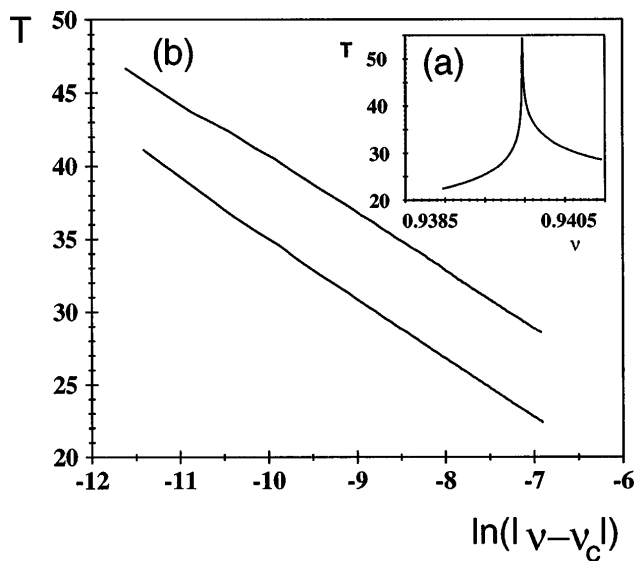

FIG. 3. Residence time T spent near the nucleation solution to (1) for $\Omega = 0.97$ ($\nu_c \simeq 0.939\,470$), (a) as a function of ν and (b) as a function of $\ln(|\nu - \nu_c(\Omega)|)$.

observe the evolution of an initial state on this surface. Then the case when the fronts are preserved corresponds to the existence of a rotating trajectory for the homogeneous system, while the case when the fronts are annihilated corresponds to the existence of a stationary solution for the homogeneous system. Because of the stabilizing effect of the torsional coupling in the extended system, the Andronov bifurcation of the nucleation solution occurs for damping $\nu_c(\Omega)$ that is slightly different from the one required for the homoclinic bifurcation of the homogeneous state $\nu_o(\Omega)$, as shown in Fig. 2. In Figs. 4(a)–4(e), we show the evolution of two colliding pulses for $\nu \sim \nu_c$ on this surface.

In the case of Neumann boundary conditions, with parameters that correspond to the preservation of colliding fronts, we observe that the fronts are reflected from the boundaries. This feature can be understood by a collision involving the front and its "image." In the case of an infinite system the preservation-annihilation transition becomes a codimension-1 bifurcation which corresponds to the coincidence of two manifolds, the stable manifold of the nucleation solution W_s and the one dimensional manifold corresponding to two fronts, which from far away move one towards the other W_{coll}.

In conclusion, we have shown that the transition from annihilation to preservation of colliding excitable fronts is an infinite-dimensional global homoclinic bifurcation à la Andronov [3]. The simple idea that the preservation of fronts is associated with a weakly first-order character of the transition from an excitable to oscillatory state, was explored in some detail in a model excitable system (1). The apparent peculiarities of (1), namely, (a) the topology associated with the periodicity of the field $\theta(x,t)$, and (b) the existence of an integrable limit of (1) are not restrictive. The geometric nature of the arguments should carry over to other systems such as the complex Ginzburg-Landau equation [13]. However, numerical and experimental studies on various systems [2,13] show that the region in parameter space where this transition occurs is small. We have found that a spatially extended system based on the

Takens-Bogdanov normal form [3] corresponding to the unfolding of a codimension-2 bifurcation exhibits this behavior [13]. Extensions currently being studied include the connection between first-order phase transition, nucleation theory, and global bifurcations, and the use of these ideas to understand the nature of excitable media in one and two dimensions.

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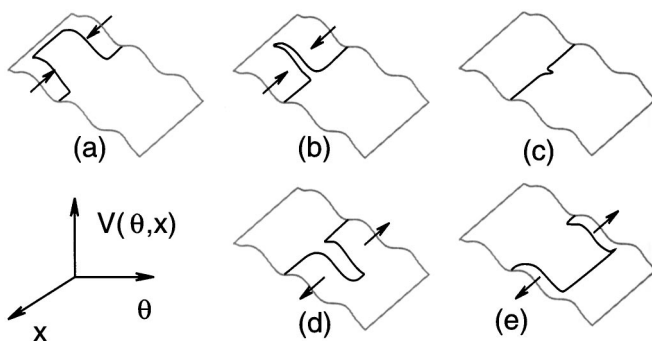[13] M. Argentina, P. Coullet, and L. Mahadevan (unpublished).

FIG. 4. Snapshots at different times of the collision and preservation of two fronts for $\nu < \nu_c$ represented as a flow on the potential surface $V(\theta, x)$. Time evolves from picture (a) to (e) the collision occurs at (c).