PHYSICAL REVIEW LETTERS

VOLUME 79 | 6 OCTOBER 1997 | NUMBER 14

# Nonequivalence between Stationary Matter Wave Optics and Stationary Light Optics

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(Received 23 May 1997)

Stationary matter wave optics and stationary light optics are equivalent when we consider one-particle systems in vacuum. We show that, in contrast, stationary optics of energy-entangled particles is completely different for matter waves and for light. This difference is illustrated comparing the two-particle interference patterns exhibited by matter waves and by light, respectively. The time-independent probability for simultaneous detection of the two massive particles or of the two photons is strikingly nonequivalent. [S0031-9007(97)04118-5]

PACS numbers: 03.75.–b, 03.65.–w, 42.50.–p

It is well known that matter waves and light show completely equivalent diffraction phenomena in stationary experiments in vacuum. There both massive-particle waves and photon waves obey the same (Helmholtz) equation. One might naively expect that the equivalence between stationary matter wave and stationary light optics is also preserved in the case when we consider multiparticle systems. In the present paper we investigate this question and we will find that this is not the case.

The present problem is not only of interest from a fundamental viewpoint but also with respect to recent experiments with entangled particles in a variety of systems (photon pairs produced by parametric down conversion [1], light-induced dissociation of molecules with laser pulses [2], simultaneous observation of atoms, and spontaneously emitted photons [3]).

The elementary multiparticle wave coming from a point source (the Green's function) together with the source distribution contain *all* information about the system considered. Thus, to examine whether or not stationary matter wave optics and stationary light optics are equivalent in general, it suffices to compare the stationary multiparticle Green's functions for matter waves and light. Throughout the paper we shall use the term particle for both massive particle and photon.

The nonstationary Green's function describes the effect of a point source localized at position $\mathbf{r}_0$ at time $t_0$ on the observation point $\mathbf{r}$ at time $t$. An elementary wave emitted by a point source and obeying the wave equation for light in vacuum [4] is a spherical shell about the point source, expanding with the radial velocity $c$ [5]

$$G^{\text{light}}(\mathbf{r}, t, \mathbf{r}_0, t_0) = \frac{\delta(|\mathbf{r} - \mathbf{r}_0|/c - (t - t_0))}{|\mathbf{r} - \mathbf{r}_0|}\,\theta(t - t_0)\,. \tag{1}$$

In contrast, the Green's function of the nonrelativistic Schrödinger equation in vacuum immediately becomes unequal zero everywhere as soon as $t$ differs from $t_0$ because here the disturbance at a point source contains all velocity components [5]

$$G^{\text{matter}}(\mathbf{r}, t, \mathbf{r}_0, t_0) = \sqrt{\frac{m}{2\pi i\hbar}}\,\frac{1}{(t - t_0)^{\frac{3}{2}}}\,e^{\frac{im|\mathbf{r}-\mathbf{r}_0|^2}{2\hbar(t-t_0)}}\,\theta(t - t_0)\,. \tag{2}$$

The step function $\theta(t - t_0)$ is required by causality.

Stationary optics implies time independence of the probability to find the particle; i.e., the probability to detect the particle at a given position in space is the same at *every* time of observation.

Now, we look for the one-particle Green's function for stationary optical experiments. We consider a point source starting at time $t_0 = -\infty$ to emit elementary nonstationary one-particle waves harmonically in time with a frequency $\omega_0$; that is, the source function is given by $e^{-i\omega_0 t_0}$. Then the effect of a point source localized at point $\mathbf{r}_0$ on the observation point $\mathbf{r}$ at time $t$ can be obtained as a superposition of all effects of elementary

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nonstationary waves emitted at different instants until the observation time $t$

$$\int_{-\infty}^{t} dt_0\, e^{-i\omega_0 t_0} G^{\text{light,matter}}(\mathbf{r}, t, \mathbf{r}_0, t_0) = \frac{e^{ik(\omega_0)|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|}\, e^{-i\omega_0 t}, \qquad (3)$$

where the dispersion relation is $k(\omega_0) = \omega_0/c$ for light and $k(\omega_0) = \sqrt{2m\omega_0/\hbar}$ for matter waves. In both cases we obtain the well-known Green's function of the Helmholtz equation. This identity of the stationary Green's function for matter waves and for light clearly implies full equivalence of stationary optics for matter waves and for light in single-particle systems.

The superposition given by expression (3) is connected with complete lack of information about the instant of emission of a particle. While thus the time of emission is completely undefined, the complementary quantity, the frequency of the one-particle wave (3), is well defined; that is, the particle has the well-defined energy $\hbar\omega_0$. For massive particles only the nonrelativistic energy is considered.

We now turn to the multiparticle case. In analogy to the one-particle case, we define stationary optics for multiparticle systems such that the *coincidence* probability is time independent; i.e., the probability to detect the particles of the system in coincidence at given positions in space is the same at *every* time of observation.

We shall now obtain stationary two-particle Green's functions for matter waves and for light. For simplicity we consider here two-particle systems, yet our considerations can easily be generalized to multiparticle systems.

In analogy to the one-particle case we consider the effect of a two-particle point source localized at position $\mathbf{r}_0$ at time $t_0$ on the observation points $\mathbf{r}_1$ of the first and $\mathbf{r}_2$ of the second particle at time $t$ [6]. We assume that after the emission the two particles propagate freely, without any interaction between them. Therefore, for the time of observation $t$ the two-particle nonstationary Green's function is the product state $G_1(\mathbf{r}_1, t, \mathbf{r}_0, t_0)G_2(\mathbf{r}_2, t, \mathbf{r}_0, t_0)$. In order to obtain the stationary two-particle Green's function we consider a point source located at $\mathbf{r}_0$ emitting elementary nonstationary two-particle waves harmonically in time with frequency $\omega_0$; that is, the source function is $e^{-i\omega_0 t_0}$, as in the one-particle case. For the infinitely long duration of emission, the resulting effect of the point source on the observation points $\mathbf{r}_1$ of the first and $\mathbf{r}_2$ of the second particle at time $t$ can be obtained as the superposition [7]

$$G_{12}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_0) = \int_{-\infty}^{t} dt_0\, e^{-i\omega_0 t_0} G_1(\mathbf{r}_1, t, \mathbf{r}_0, t_0) \times G_2(\mathbf{r}_2, t, \mathbf{r}_0, t_0)\,. \qquad (4)$$

The superposition (4) is connected with complete lack of information about the instant of emission of the pair. While thus the time of emission is completely undefined, the complementary quantity, the frequency of the two-particle wave (4), is well defined; that is, the pair has the well-defined *total* energy $\hbar\omega_0$. Yet we have no information which particle takes which part of the total energy. This is most directly seen when we rewrite the stationary two-particle Green's function (4) in terms of its Fourier components as [8]

$$G_{12}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_0) = \int d\omega\, \frac{e^{ik(\omega)|\mathbf{r}_1-\mathbf{r}_0|}}{|\mathbf{r}_1 - \mathbf{r}_0|} \times e^{-i\omega t}\, \frac{e^{ik(\omega_0-\omega)|\mathbf{r}_2-\mathbf{r}_0|}}{|\mathbf{r}_2 - \mathbf{r}_0|}\, e^{-i(\omega_0-\omega)t}. \qquad (5)$$

Thus the effect of a stationary two-particle point source is a superposition of all product states of single-particle stationary Green's functions for the first and second particles with such combination of the wave numbers that the sum of the energies of the two particles is always $\hbar\omega_0$. This clearly is a highly entangled state. While the energy of the pair is well defined, we have complete lack of information about the energy of each individual particle. It is also clear that the superposition (5) now depends sharply on the specific dispersion relation $k(\omega)$.

We emphasize that although at *every* instant of emission the point source emits the *nonentangled* elementary two-particle wave $G_1(\mathbf{r}_1, t, \mathbf{r}_0, t_0)G_2(\mathbf{r}_2, t, \mathbf{r}_0, t_0)$ the resulting elementary stationary two-particle wave (5) is a highly *entangled* state. Also, it is obvious that the coincidence probability obtained from the two-particle wave function (5) is time independent. It can be shown that even when the particles are observed at constant time difference the coincidence probability is again time independent.

The stationary two-particle Green's function for light obtained from Eq. (4) by direct integration is

$$G_{12}^{\text{light}}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_0) = \frac{c\,\delta(|\mathbf{r}_1 - \mathbf{r}_0| - |\mathbf{r}_2 - \mathbf{r}_0|)}{|\mathbf{r}_1 - \mathbf{r}_0|\,|\mathbf{r}_2 - \mathbf{r}_0|} \times e^{i\frac{\omega_0}{c}(|\mathbf{r}_2-\mathbf{r}_0|-ct)}. \qquad (6)$$

Thus if one photon is detected at point $\mathbf{r}_1$, the probability to find simultaneously another photon is unequal zero only on the spherical shell centered at the common point source with the radius $|\mathbf{r}_1 - \mathbf{r}_0|$. This can be seen as a consequence of the fact that the two photons produced simultaneously both propagate with the constant velocity of light $c$. By detection of the first photon at point $\mathbf{r}_1$ at time $t$ we obtain information about the instant of emission of both photons: $t_0 = t - \frac{|\mathbf{r}_1-\mathbf{r}_0|}{c}$.

The stationary two-particle Green's function for matter waves obtained from the integral (4) is

$$G_{12}^{\text{matter}}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_0) \propto \frac{e^{i\sqrt{\frac{2\omega_0}{\hbar}(m_1|\mathbf{r}_1-\mathbf{r}_0|^2+m_2|\mathbf{r}_2-\mathbf{r}_0|^2)}}}{(m_1|\mathbf{r}_1 - \mathbf{r}_0|^2 + m_2|\mathbf{r}_2 - \mathbf{r}_0|^2)^{5/4}}\, e^{-i\omega_0 t}. \qquad (7)$$

Since vacuum is dispersive for matter waves, detection of one particle at point $\mathbf{r}_1$ at time $t$ gives no information about the instant of emission of the pair. However, for the point $\mathbf{r}_0$ and the time $t_0$ of emission of the pair, the classical propagation velocity of the detected particle is $\frac{\mathbf{r}_1 - \mathbf{r}_0}{t - t_0}$. This further implies the energy $\hbar\omega_1 = \frac{m_1|\mathbf{r}_1 - \mathbf{r}_0|^2}{2(t-t_0)^2}$ of the detected particle. Then, the energy conservation condition $\hbar\omega_0 = \hbar\omega_1 + \hbar\omega_2 = \frac{m_1|\mathbf{r}_1 - \mathbf{r}_0|^2}{2(t-t_0)^2} + \frac{m_2|\mathbf{r}_2 - \mathbf{r}_0|^2}{2(t-t_0)^2}$ can be seen as implying a relation between instants $t_0$ of emission of the pair and corresponding points $\mathbf{r}_2$ of detection of the second particle for every given point $\mathbf{r}_1$ and time $t$ of detection of the first particle. While thus the instant of emission of the second particle is not well defined, its Green's function is a superposition over all instants of emission $t_0$, each instant implying a different but definite position $\mathbf{r}_2$ for coincidence detection of the second particle.

To conclude, in contrast to the one-particle case, the stationary multiparticle Green's functions are analytically completely *different* for matter waves and for light [9]. In order to illustrate this difference we now examine two-particle interference patterns arising from the superposition of stationary two-particle waves coming from two spatially distant coherent point sources. This interference pattern is conditional; that is, one does not look at either particle *separately*, but monitors the arrival positions of two particles *in coincidence*. The interference pattern is stationary. That is, one can record coincidences at *any* time; the coincidence probability to find particles at fixed positions does not depend on the instant of observation. Two spatially distant coherent two-particles sources could be, for example, a pair of small spatially distant down-conversion crystals pumped by the same laser or a dissociating molecule in a superposition of two spatially distant states.

We assume that point sources $P$ and $Q$ located on the $y$ axis at the points $y = \pm a$ (Fig. 1) simultaneously emit two particles in the stationary state (4) of total energy $\hbar\omega_0$. We shall observe interference patterns formed by one particle along the line $L$ parallel to the $y$ axis at the distance $x_0$ conditional on detection of the other particle at the same time by detector $D_1$ fixed at position $\mathbf{r}_1$ on line $L$.

The interference pattern, characteristic of the specific Green's function, results from the superposition of two possibilities: either both particles are emitted at the source point $P$ or both particles are emitted at the source point $Q$:

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = G_{12}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_P) + G_{12}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_Q). \tag{8}$$

For light the probability for coincidence detection is [10]

$$|\psi^{\text{light}}(\mathbf{r}_1, \mathbf{r}_2)|^2 \propto \frac{1}{x_0^2}\Big\{\delta^2((y_1 - y_2)(y_1 + y_2 + 2a)) + \delta^2((y_1 - y_2)(y_1 + y_2 - 2a)) + 2\delta((y_1 - y_2)(y_1 + y_2 + 2a))\delta((y_1 - y_2)(y_1 + y_2 - 2a))\cos\left(\frac{\omega_0}{c}\frac{a}{x_0}y_2\right)\Big\} \tag{9}$$

in the Fraunhofer limit. If one photon is caught at point $\mathbf{r}_1$, the other photon generated simultaneously at the common point source could at the same time of observation be found only on two spherical shells (Fig. 1) centered at the point sources $P$ and $Q$ with the radii $|\mathbf{r}_1 - \mathbf{r}_P|$ and $|\mathbf{r}_1 - \mathbf{r}_Q|$, respectively. The intersection points of the line $L$ with the sphere centered at point $P$ are located at $y_2 = y_1$ and $y_2 = 2a - y_1$. Similarly, the intersection points of the line $L$ with the sphere centered at point $Q$ are located at $y_2 = y_1$ and $y_2 = -2a - y_1$. It is important to notice that interference [the third term in Eq. (9)] occurs only at points where we have no information from which source the second photon comes, i.e., at the intersection points of two spherical shells. This occurs only at the point $y_1 = y_2$ on the line $L$.

The conditional pattern exhibited by massive particles is totally different from the pattern exhibited by photons. Thus if the first massive particle is caught at the point $\mathbf{r}_1$, the other particle forms conditional fringes of Young's type along line $L$ according to

$$|\psi^{\text{matter}}(\mathbf{r}_1, \mathbf{r}_2)|^2 \propto \cos^2\left[\sqrt{\frac{2\omega_0}{\hbar}}\,\frac{m_1 y_1 + m_2 y_2}{\sqrt{m_1 + m_2}}\,\frac{a}{x_0}\right], \tag{10}$$

within the Fraunhofer approximation [11].

Assuming for the stationary two-particle Green's function a product state $G_{12}(\mathbf{r}_1, \mathbf{r}_2, t, \mathbf{r}_0) = \frac{e^{ik_1|\mathbf{r}_1 - \mathbf{r}_0|}}{|\mathbf{r}_1 - \mathbf{r}_0|}\frac{e^{ik_2|\mathbf{r}_2 - \mathbf{r}_0|}}{|\mathbf{r}_2 - \mathbf{r}_0|}$ Horne [12] reported the solution

$$|\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 \propto \cos^2\left[(k_1 y_1 + k_2 y_2)\frac{a}{x_0}\right]. \tag{11}$$

This is also Young's pattern. While entangled in momentum, Horne's particle having well-defined wave numbers $k_1$ and $k_2$ are not entangled in energy. The Green's function used by Horne follows from our Green's function (5) when a filter of infinitely narrow bandwidth is placed before one of the detectors. Because of the energy conservation, the filtering of one of the particles also defines ("nonlocally") the frequency of the second particle. Obviously, the conditional interference patterns exhibited by massive particles and by photons are now equivalent.

The existence of limiting cases, of complete difference between conditional patterns exhibited by matter waves and light (when broadband detectors are used) and complete equivalence between two conditional patterns (when

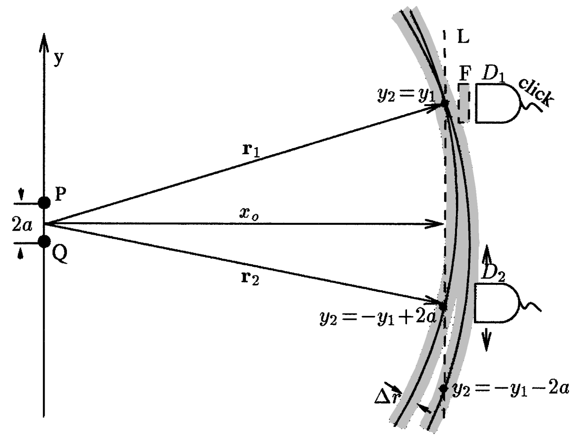


FIG. 1. Scheme for observation of two-particle interference patterns. The stationary point sources $P$ and $Q$ emit coherently two-particle waves. Detector $D_1$ is fixed at position $\mathbf{r}_1$ on line $L$. Detector $D_2$ is moved along the line $L$ to detect the distribution of the second particle *conditional* on detection of the first particle in $D_1$. In the case of light, if no filter is inserted before $D_1$ the second photon can be found only on two spherical shells of infinitely small width. Interference occurs only at their intersections. If a filter $F$ of bandwidth $\sigma$ is inserted in front of $D_1$ the two spherical shells of significant conditional probability have a finite width $\Delta r = c/\sigma$. In the intersection region conditional Young's fringes arise. Pairs of massive particles exhibit a different kind of Young's fringes in the whole Fraunhofer region already without filtering.

a filter of infinitely narrow resolution is inserted before one of the detectors), suggests that there are intermediate cases. Thus, if a filter of bandwidth $\sigma$ is placed before one of the detectors, the space region where conditional probability for the second photon significantly differs from zero lies on two spherical shells of finite width $\Delta r = c/\sigma$ (Fig. 1). The solution at the intersection area of the two spherical shells contains the interference term, resulting in a Young's pattern. For an infinitely narrow filter $\sigma \rightarrow 0$ the region with Young's fringes spreads over the whole Fraunhofer region $\Delta r \rightarrow \infty$, as predicted by Eq. (11). For matter waves, with a decrease of the filter bandwidth the Young's pattern of Eq. (10) also continuously transforms into the one given by Eq. (11). The narrower the bandwidth of the inserted filter, the larger the area of resemblance becomes between the matter waves and light.

Stationary matter wave optics and stationary light optics are equivalent when we consider one-particle systems in vacuum; that is, the stationary one-particle Green's functions for matter waves and for light are equivalent. In contrast, the stationary Green's function for energy-entangled particles is completely different for matter waves and for light. Consequently, in general the (time-independent) probability for coincident detection of energy-entangled particles at fixed positions in space is different for matter waves and for light.

This work has been supported by the Austrian Science Foundation FWF, Project No. S6504 and by the U.S. National Science Foundation under Grant No. 97-22614.

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[4] For many diffraction phenomena it suffices to describe light as a scalar field. See, for example, M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1980), pp. 387–392; A. Lagendijk and B. A. van Tiggelen, Phys. Rep. **270**, 143 (1996).
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[6] The generalized cases of two particles generated at different space points and at different but mutually *dependent* times of emission or observed at different times lead to equivalent conclusions.
[7] This two-particle Green's function is symmetric for identical particles. It therefore applies to bosons with symmetric internal states or fermions with antisymmetric internal states. It also applies to distinguishable particles where symmetrization is not needed.
[8] In principle, the negative frequency contributions to the integral (5) are unphysical. For simplicity we retain these contributions. Constraining the integration interval in Eq. (5) to $[0, \omega_0]$ would be physically equivalent to the existence of a finite time uncertainty in the instant of emission of one particle with regard to the instant of emission of the other particle.
[9] A referee observed that similar results could have been obtained for a system of Klein-Gordon particles, which might be relevant as a model for entangled systems in nuclear physics. The referee noticed that our results for photons and massive particles can then be recovered in the ultrarelativistic $k \gg mc$ and nonrelativistic $k \ll mc$ limits, respectively. The Klein-Gordon particles hence interpolate smoothly between the behaviors analyzed by us.
[10] The approach presented in the present paper differs from the usual second quantization formalism where one considers field operators rather than the quantum states defined on the multiparticle configuration space. The two-photon interference pattern predicted by Eq. (9) would correspond to an intensity correlation function $\langle E(\mathbf{r}_1)E(\mathbf{r}_2)E^*(\mathbf{r}_2)E^*(\mathbf{r}_1)\rangle$ of the electric field operator $E(\mathbf{r})$ in the second-quantized formalism. The mean value is taken with respect to the appropriate field state.
[11] One may consider also multiparticle systems consisting of both photons and massive particles. The corresponding stationary Green's function differs from the stationary

Green's function for both matter waves and light. Spatially distant coherent sources of a photon and a massive particle could be realized by a decaying atom in superposition of two spatially distant states. The conditional interference pattern is again of Young's type.

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