Nature of Resistive Transitions in Josephson Networks in a Magnetic Field

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We study the depinning of the vortices induced in a Josephson network by a magnetic field. A vortex crystal can be pinned when it can be made commensurate with the network potential by rotation; this can occur only when the number of vortices per unit cell p/q is constructed from special values of the integers p and q. For these cases the transition can be of roughening type; however, melting of the vortex crystal is an important competing process that changes the nature of the transition for $q < 16$ (square lattice) or $q < 12$ (hexagonal lattice). For the other values of p and q there can be a spontaneous deformation of the vortex crystal to a commensurable structure if the network potential is sufficiently large. [S0031-9007(97)04082-9]

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The low energy states of a Josephson network are well characterized by the locations of its vortices [1]; in particular, its resistance can be regarded as a consequence of vortex motion. This maps the Josephson network problem onto a model of point vortices interacting with a two-dimensional Coulomb potential $V(r) = -2\pi J \ln(r/a)$, where J is the Josephson coupling constant. The network provides a periodic potential which attempts to confine the vortices to lattice sites, and which can pin the vortex crystal when they are commensurate. We will assume this potential to have either square or hexagonal symmetry, with lattice constant a and unit cell area $\mathcal{A}$. To have commensuration the number of vortices per unit cell of the network must be a rational number: $n\mathcal{A} = p/q$, where p and q are mutually prime. The density of vortices is set by the applied field, and so the resulting system is incompressible.

Some attempts to understand the resistive transition in the Josephson network make use of the phase representation. The molecular-field approximation [2] and linearized Landau-Ginzburg theory [3] indicated a rich phase diagram. The model has been extensively studied in Monte Carlo simulations [4]. These works are somewhat ambiguous, because we would expect the transitions in a two-dimensional system with a continuous symmetry to be of Kosterlitz-Thouless [5] type, which is very poorly represented by mean-field theories; this would also entail correlations that decay algebraically with distance, implying very large finite-size corrections to numerical simulations.

The resistive transition has also been discussed in terms of the melting of the vortex crystal [1]. The existence of distinct ground states having different registry of the crystal suggests a relationship to the Potts models; however, most finite domains will have net vorticity [6] and thus have long-range interactions which again puts the problem in the Kosterlitz-Thouless class.

The common feature of these previous approaches is that they have put the particles on a lattice and then observed the effect of the vortex-vortex interactions. The present approach will instead treat the periodic potential as being a perturbation of the vortex crystal.

At zero temperature, the vortices form a crystal of lattice vectors $\{\vec{R}\}$ with primitive vectors of length b (determined by the vortex density). Thermal fluctuations give rise to displacements $\vec{u}(\vec{R})$ away from these sites which can destroy long-range order, but orientational order can survive up to temperatures sufficiently high that dislocations become probable [7]. Within the harmonic approximation, the interactions are represented by the elastic energy $\frac{1}{2}\mu\sum_{i,j=x,y}|\partial u_i/\partial x_j|^2$, where μ is the shear modulus. Only one elastic constant enters, because the vortex liquid is incompressible; correspondingly, we consider only displacement fields that are divergenceless. The shear modulus is related to the applied field by [8] $\mu = \pi JB/4\Phi_0$ (where Φ_0 is the flux quantum) and thus is proportional to the density of vortices $n = B/\Phi_0$.

The system is described by the Hamiltonian

$$H = \int d^2r \tfrac{1}{2}\mu\Sigma_{i,j}|\partial u_i/\partial x_j|^2 - \Sigma_R V(\vec{R} + \vec{u}(\vec{R}))\,. \tag{1}$$

The potential V tries to localize the vortices to the network's dual lattice (the centers of the plaquettes defined by the superconducting islands), and is periodic with a set of lattice vectors $\{\vec{R}'\}$. Using the Poisson sum rule to represent the sum over $\{R\}$ puts (1) into the form

$$H = \int d^2r\{\tfrac{1}{2}\mu\Sigma_{i,j}|\partial u_i/\partial x_j|^2 - \Sigma_{G,G'}nV_{G'}\cos[(\vec{G} - \vec{G}')\cdot\vec{r} + \vec{G}\cdot\vec{u}(\vec{r})]\}\,, \tag{2}$$

where $\vec{G}$ and $\vec{G}'$ are the reciprocal lattice vectors corresponding to the two lattices $\vec{R}$ and $\vec{R}'$, and $V_{G'}$ is a Fourier component of $V(\vec{R})$. The two sets $\{\vec{R}\}$ and $\{\vec{R}'\}$ (or $\{\vec{G}\}$ and $\{\vec{G}'\}$) refer, respectively, to vortex crystal and to the periodic potential provided by the underlying network: they are quite distinct. We will study the conditions under which they have elements in common, and whether the potential then succeeds in pinning the crystal. Some parts of this problem have been discussed previously in other contexts [9,10].

At high temperatures the vortices can move, which gives rise to dissipation; at low temperatures the vortex crystal can be pinned and then the network is superconducting. At intermediate temperatures it is possible that the vortex crystal exists, but is not pinned; then a current will cause it to move, giving rise to flux flow resistance. Thus three different effects appear to be relevant to understanding the resistive transition:

(1) The vortex crystal could be incommensurate with the periodic potential. This can happen even for rational $n\mathcal{A}$—in particular it occurs for the (expected) hexagonal vortex crystal and a periodic potential of square symmetry. In this case the network potential may induce a shear deformation of the vortex crystal, making it commensurable. The temperature scale for this deformation depends on the strength of the periodic potential as well as on the amount of distortion required, and thus can be distinct from the resistive (vortex depinning) transition. The existence of this transition to commensuration allows us to study the cases that the vortex crystal has symmetry other than hexagonal.

(2) When the vortex crystal is commensurate with the network, the study of pinning reduces to the question of the relevance (in the renormalization sense) of the network potential. The transition temperature depends most strongly on the shear modulus of the vortex crystal and on the wavelength of the commensurate Fourier component of the network potential, and only weakly on the strength of the potential.

(3) The vortex crystal could melt, due to the unbinding of dislocations. The melting temperature is determined by the shear modulus of the vortex crystal and the spacing between vortices (which is the Burger's vector of the dislocations).

These mechanisms are all described by theories of Kosterlitz-Thouless type; their relative importance depends on the specific rational value of $n\mathcal{A}$. Of the three, the induction of commensuration by a spontaneous shear is of secondary interest, because it need not be directly involved in the resistive transition. The periodic potential and the thermal excitation of dislocations play antithetical roles, however; the potential induces crystalline order while dislocations disrupt it. We will first discuss melting in the absence of the periodic potential and pinning in the absence of dislocations, and subsequently discuss the competition between them.

Melting of the vortex crystal.—Dislocations are point defects in a two-dimensional crystal. At lowest temperatures all dislocations are bound into pairs of small separation by their strain fields. The presence of dislocations decreases the shear modulus from its low temperature value, however, which decreases the barrier to thermal excitation of the pairs. The shear modulus drops discontinuously to zero at the melting temperature, which is related to the shear modulus by the universal relation [7]

$$T_{\mathrm{melt}} = \mu(T_{\mathrm{melt}})b^2/4\pi\,. \tag{3}$$

At zero temperature, $\mu(0)b^2$ is independent of the field strength; $\mu(T_{\mathrm{melt}})$ is less than $\mu(0)$ by an amount that depends on the dislocation core energy [7].

Pinning of the vortex crystal.—In renormalization language we are asking whether any of the Fourier components of the network potential is a relevant perturbation. This problem has been discussed in the context of films absorbed to crystalline substrates [7]. There are two conditions that the potential must obey:

(i) There can be no explicit r dependence of the argument of the cosine in Eq. (2); thus the only relevant terms are those for which $\vec{G} = \vec{G}'$.

(ii) The temperature must be sufficiently low. The effect of a commensurate potential is to increase the shear modulus, which suppresses the thermal fluctuations in $\{\vec{u}\}$; as the temperature is lowered the shear modulus jumps to infinity at the pinning temperature, which is related to the critical value of the shear modulus by

$$T_{\mathrm{depin}} = 16\pi\mu(T_{\mathrm{depin}})/G^2. \tag{4}$$

In deriving this expression it is assumed that the longitudinal modes of the $\vec{u}$ field have finite frequency in the long-wavelength limit (corresponding to the incompressibility of the vortex crystal) and thus are suppressed.

We will consider first the case that the network is a square lattice of spacing a, and that the reference state for the vortices is also a square lattice of lattice spacing $b = \sqrt{q/p}\,a$, as determined by the density. These two structures can share reciprocal lattice vectors for certain choices of p and q, as will now be shown. Let the primitive reciprocal lattice vectors for the vortex reference crystal be $\vec{G}_1$ and $\vec{G}_2$, and let the general reciprocal lattice vector for the network be $\vec{G}' = 2\pi(r,s)/a$. Commensuration is possible if there will be integers r, s, t, and u such that $t\vec{G}_1 + u\vec{G}_2 = 2\pi(r,s)/a$. Comparison of the magnitudes shows that $4\pi^2(r^2+s^2)/a^2 = 4\pi^2(t^2+u^2)/b^2 = 4\pi^2(t^2+u^2)p/qa^2$; the minimal solution is then described by the solutions to $t^2+u^2 = q, r^2+s^2 = p$. These equations have solutions only when p and q are members of the set [11] $1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, \ldots$. For the allowed p's and q's the solution to $\vec{G} = \vec{G}'$ with smallest magnitude gives $G^2 = 4\pi^2 p/a^2$, and thus

$$T_{\mathrm{depin}} = 4\mu(T_{\mathrm{depin}})a^2/\pi p\,. \tag{5}$$

In the limit of weak periodic potential the critical value $\mu(T_{\mathrm{depin}})$ is well approximated by its zero-temperature value [12] (proportional to $n = p/qa^2$), and thus the real prediction of Eq. (5) is $T_{\mathrm{depin}} \propto 1/q$. Since q is a highly discontinuous function of the vortex density, this predicts a very erratic dependence of the resistive transition temperature on magnetic field [13].

As an example, consider $na^2 = 2/5$: the square array of vortices of spacing $b = \sqrt{5}a/\sqrt{2}$ can be rotated so that its primitive vectors are $\vec{b}_1 = (\frac{3}{2}, \frac{1}{2})a$ and $\vec{b}_2 = (\frac{1}{2}, -\frac{3}{2})a$, which is now commensurate with the network; although the vortices do not all lie on lines of spacing a parallel to the (1,0) direction of the network, they do lie on lines of spacing $a/\sqrt{2}$ in the (1,1) direction, corresponding to a common reciprocal lattice vector of magnitude $2\sqrt{2}\pi/a$. In contrast, for $na^2 = 1/3$, a square array of vortices of spacing $\sqrt{3}a$ cannot be made commensurate with the network at all.

The case of a hexagonal vortex ground state and a hexagonal network potential (again of spacing a—note that this entails a honeycomb pattern of superconductors) is similar. The reciprocal lattice vectors $\vec{G}'$ have the form $\vec{G}' = (2\pi/\sqrt{3}a)(\sqrt{3}r, 2s + r)$, where r and s are integers. We again seek integers r, s, t, and u such that $t\vec{G}_1 + u\vec{G}_2 = 2\pi(r, s)/a$, where $\vec{G}_1$ and $\vec{G}_2$ are the primitive reciprocal lattice vectors for the vortex reference crystal, described by $|\vec{G}_1|^2 = |\vec{G}_2|^2 = 2\vec{G}_1 \cdot \vec{G}_2 = 16\pi^2/3b^2$. The minimal solution is then described by the solutions to $t^2 + tu + u^2 = q, r^2 + rs + s^2 = p$, which have solutions only when p and q are members of the set [14] $1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 25, \ldots$ For the allowed p's and q's the solution to $\vec{G} = \vec{G}'$ with smallest magnitude gives $G^2 = 16\pi^2 p/3a^2$, so that

$$T_{\mathrm{depin}} = 3\mu(T_{\mathrm{depin}})a^2/\pi p \,. \tag{6}$$

This difference from the square lattice comes about because the important dimension is the height of the triangles (the distance between lines of vortices), rather than the length of their sides.

Induction of commensuration.—The theory described above requires that the vortex crystal and potential are commensurable. It would seem that this greatly curtails the applicability of this theory, since in the absence of a periodic potential, the vortex crystal has hexagonal symmetry which is never commensurable with a periodic potential of square symmetry, and is not commensurable with a potential of hexagonal symmetry for many values of $n\mathcal{A} = p/q$. The different symmetry between vortex crystal and network potential would suggest a discontinuous transition in all such cases. However, even when the potential fails to pin the crystal it can induce a spontaneous distortion of the vortex crystal to a geometry which is commensurable.

Consider the pinning of a vortex crystal by a general periodic potential in the presence of a shear stress that acts on the vortex crystal and can deform it away from hexagonal symmetry. Rather than specify the stress, however, let us specify the resulting strain and then compute the stress from how the free energy changes with this strain. For structures that are not too far from hexagonal the resulting crystal is stable against small displacements and has a sensible elastic theory, which in the harmonic approximation has the same shear modulus as did the undeformed hexagonal crystal. For an appropriate strain the vortex crystal is commensurable (in fact there are many ways to do this when q is large), and its pinning can be studied by a small generalization of the theory described above. The same renormalization theory that calculates the temperature dependence of the shear modulus can also be used to determine the free energy [15], revealing that this is lowered by the effect of the periodic potential. The size of this decrease is determined by the amplitude of the periodic potential and may well compensate for the cost in elastic energy in making the original strain, which can be small when the unit cell is large: this structure is then stable relative to the hexagonal crystal. Thus for large enough q there will always be a continuous pinning transition.

The theory for T_{depin} is more complicated in this new context, because now there is no rotational symmetry and the lattice vectors describing the vortex crystal unit cell are different. The shear modulus may be slightly altered, but the change will be small when the ground state is nearly hexagonal (which it always is for large q). The deformed vortex crystal will have a new set of reciprocal lattice vectors but the commensuration is still described by Eq. (2), and the relevant terms are again those for which $\vec{G} = \vec{G}'$: the essential feature will continue to be the identification of the relevant periodicity of the network potential, which is unchanged. Then Eq. (4) implies that the depinning temperature is given by equations similar to Eqs. (5) or (6): for example, with a square symmetry network $T_{\mathrm{depin}} = 4\mu(T_{\mathrm{depin}})a^2/\pi k$ where k is again an integer chosen from the list $1, 2, 4, 5, \ldots$ but now it is not clear how it is determined by the magnetic field.

Here and in the case of a pinning potential of low symmetry there is the possibility of an unusual phase in which only one reciprocal lattice vector is relevant [16]. It would be superconducting in the direction perpendicular to the relevant G, but have finite resistance in all other directions.

Competition between melting and depinning.—Combining Eqs. (3), (5), and (6), and neglecting renormalization of μ, we find [7]

$$T_{\mathrm{depin}}/T_{\mathrm{melt}} = 16/q \qquad \text{(square lattice)}\,, \tag{7}$$

$$T_{\mathrm{depin}}/T_{\mathrm{melt}} = 12/q \qquad \text{(hexagonal lattice)}\,, \tag{8}$$

which seems to imply that the depinning transition of Kosterlitz-Thouless (roughening) type will not be visible for $q < 12$ (which includes almost every case that has ever been studied), since the vortex crystal melts at a lower temperature, which depends only on the vortex density and not on the integers p or q separately. For small q the resistive transition connects the pinned crystal to the unpinned vortex fluid and is first order; only for large enough q is there a floating crystal phase between the superconductor and the vortex liquid [4,17].

The statistical mechanics of a system having dislocations, a periodic potential, and an elastic constant is modeled by renormalization flow equations having the general form

$$dV/d\ell = 2(1 - K_{\mathrm{depin}}/K)V\,, \qquad (9)$$

$$dy/d\ell = 2(1 - K/K_{\mathrm{melt}})y\,, \qquad (10)$$

$$dK/d\ell = \alpha V^2 - \beta y^2, \qquad (11)$$

where $K = \mu a^2/T$ is a dimensionless measure of the shear modulus, V is the amplitude of the relevant Fourier component of the network potential, y is the dislocation fugacity, and ℓ (the usual renormalization group parameter) sets the length scale. In the absence of dislocations ($y = 0$), the remaining equations describe a depinning transition associated with the fixed point of Eq. (9), where K has the universal value K_{depin} [whose value is implied by Eqs. (5) or (6)]; in the absence of a periodic potential ($V = 0$) the melting transition corresponds to the fixed point of Eq. (10), where $K = K_{\mathrm{melt}}$ [whose value is implied by Eq. (4)]; α and β are model-dependent constants. When $K_{\mathrm{depin}}/K_{\mathrm{melt}} > 1$, the melting temperature is above the depinning temperature; and for $K_{\mathrm{depin}} > K > K_{\mathrm{melt}}$ both V and y decrease exponentially under renormalization, indicating the presence of a depinned vortex crystal. According to Eq. (7) this is what happens for $q > 16$ in square geometry.

In the opposite case $K_{\mathrm{depin}}/K_{\mathrm{melt}} < 1$, the melting temperature is below the pinning temperature, and now in the interval $K_{\mathrm{melt}} < K < K_{\mathrm{depin}}$, both y and V are increasing exponentially. The flow eventually goes to $V = \infty$ (pinned) or $y = \infty$ (melted), depending on the initial conditions; the pinned crystal superconductor is separated from the unpinned vortex fluid by a first-order phase transition.

The experimental case corresponds to a large pinning potential and a large core energy for dislocations (and thus to a small initial value for y). Now it is possible that the issue of the relevance of the pinning potential has already been resolved (V has renormalized to zero or μ has renormalized to a large value) on a scale ℓ for which the dislocation fugacity is still small; only very close to the depinning temperature will the renormalization proceed to sufficient length scales that the dislocations play a role. Thus in finite-sized or slightly disordered systems the transition may closely resemble Kosterlitz-Thouless type.

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