Solitary Waves in Quadratically Nonlinear Resonators

C. Etrich, U. Peschel, and F. Lederer
Institut für Festkörpertheorie und Theoretische Optik,
Friedrich-Schiller-Universität Jena,
Max-Wien-Platz 1, 07743 Jena, Germany
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We identify two-dimensional stable and unstable bright solitary waves or localized structures in a planar resonator with a quadratically nonlinear medium driven by a field at the fundamental frequency only. These waves are extremely localized while the nonlocal interaction between the fundamental and second harmonics prevents a collapse. To a certain extent they can be regarded as residuals of coexisting hexagon patterns. [S0031-9007(97)04100-8]

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Bright solitary waves or localized structures are among the most fascinating objects in nonlinear physics. They introduce some discrete or particlelike behavior into otherwise continuous systems. In nonlinear optical systems stable solitary waves are found as rather robust self-organized light distributions (see, for example, [1]). Especially, in externally driven optical resonators a proper excitation of the system may lead to the formation of a discrete number of well-defined localized structures on a low intensity background defined by the incident field. This could be the basis of a future all-optical signal processing and storage where one localized structure corresponds to one bit. So far such structures were identified in the transmitted field of resonator geometries with intensity dependent nonlinearities. Examples are two-dimensional localized structures in cavities with saturable focusing [2] or saturable absorbing media [3].

The question arises of whether stable localized structures exist also in resonators with a quadratically nonlinear material. In conservative systems, e.g., for the field evolution in a planar waveguide or bulk material, one- and two-dimensional solitary waves could be identified as symbiotic structures of the fundamental and second harmonics [4–11]. We are interested in similar structures in a planar resonator. Resonators with quadratically nonlinear media are well established for frequency up- and down-conversion [12,13]. Here we do not focus on efficient frequency conversion rather than on the spatiotemporal evolution of both waves. Thus, in contrast to most of the earlier investigations, we assume a driving field at the fundamental frequency.

We consider a planar resonator with a quadratically nonlinear medium. The frequencies of the incident fundamental field and the generated second harmonic should be close to resonances. Thus the well-established modal theory can be applied [14,15], which simplifies the analysis considerably compared to approaches based on forward and backward propagating fields. The appropriately scaled evolution equations for the transmitted fields A_1 and A_2 of the fundamental and second harmonics are derived as

$$i\,\frac{\partial A_1}{\partial T} + \frac{\partial^2 A_1}{\partial X^2} + \frac{\partial^2 A_1}{\partial Y^2} + (\Delta_1 + i)A_1 + A_1^* A_2 = E\,,$$
$$i\,\frac{\partial A_2}{\partial T} + \alpha\left(\frac{\partial^2 A_2}{\partial X^2} + \frac{\partial^2 A_2}{\partial Y^2}\right) + (\Delta_2 + i\gamma)A_2 + A_1^2 = 0\,, \tag{1}$$

where Δ_1 and Δ_2 are the detunings of the two fields from the corresponding resonances scaled in terms of the resonance width at the fundamental frequency. Though they have nothing to do with the common phase mismatch, they play a similar role in Eqs. (1). The time T is scaled in terms of the photon lifetime at the fundamental frequency and the spatial variables X and Y in terms of the square root of the product of the fundamental wavelength, velocity of light, and the photon lifetime. Thus γ is the ratio of the photon lifetimes and α half the ratio of the refractive indices corresponding to the fundamental and second harmonics. Throughout the analysis we assume $\alpha = 1/2$, which is a reasonable approximation for realistic configurations. The input field of the fundamental is E where an arbitrary phase can be transformed away. The fields are scaled in terms of the effective nonlinear coefficients arising from the second-order susceptibilities and the overlap integrals entering into the modal theory [15]. The absolute value of the overlap integrals depends critically on the phase mismatch between the fundamental and second harmonics.

For large absolute values of the detuning of the second harmonic different signs result in effective focusing ($\Delta_2 < 0$) or defocusing ($\Delta_2 > 0$) behavior. This is evident from neglecting the derivatives in the second of Eqs. (1) for large Δ_2 and substituting for A_2 in the first of Eqs. (1) leading to a cubic term there.

We are interested in localized structures (bright solitary waves evanescent to a finite background of plane wave solutions) of Eqs. (1). As a prerequisite for such structures we look for bistability of the homogeneous or plane wave solutions of Eqs. (1) where the lower branch is stable with respect to arbitrary (i.e., homogeneous and spatially modulated) perturbations, and the upper branch is unstable

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with respect to spatially modulated perturbations of the plane wave solutions. The plane wave solutions are obtained equating the derivatives in Eqs. (1) to zero [16] which yields for the moduli of the fields

$$[|A_{10}|^4 + 2(\gamma - \Delta_1\Delta_2)|A_{10}|^2 + (\Delta_1^2 + 1)(\Delta_2^2 + \gamma^2)]|A_{10}|^2 = (\Delta_2^2 + \gamma^2)E^2,$$
$$|A_{10}|^2 = |A_{20}|\sqrt{\Delta_2^2 + \gamma^2}\,. \tag{2}$$

The above polynomial for $|A_{10}|^2$ has three real solutions for certain parameter ranges if

$$\frac{|\Delta_2|(|\Delta_1| - \sqrt{3})}{\sqrt{3}|\Delta_1| + 1} > \gamma, \qquad \Delta_1\Delta_2 > 0\,, \tag{3}$$

i.e., both detunings must have equal signs, which we assume negative for the solutions under consideration to exist. For large negative Δ_2 this corresponds to the effective focusing case, as pointed out above. By means of a linear stability analysis with spatially homogeneous perturbations of the plane wave solutions we find that there is bistable behavior, with the plane wave solutions destabilizing and stabilizing at a pair of limit points. A typical situation for negative Δ_1 is depicted in Fig. 1 where the loci of critical points from the linear stability analysis of plane wave solutions are displayed in the $(\Delta_2, |A_{10}|^2)$ plane [which is equivalent to the (Δ_2, E) plane, compare first of Eqs. (2)]. For spatially homogeneous perturbations, i.e., pertubations with infinite period, they are stable in domains I, IV. This stability is referred to as homogeneous stability. The plane wave solutions are homogeneously unstable via limit points in domain II and via Hopf bifurcations in domain III (as to Hopf bifurcations, cf. [17]). Taking into account spatially modulated perturbations, i.e., with finite period, the modulational instability sets in at the bold line. Plane wave solutions which are homogeneously stable destabilize there with finite period. This leads to the formation of hexagon patterns (see below). Otherwise, the modulational instability sets in at the first limit point with infinite period (continuation of bold line), leaving the plane wave solutions modulationally unstable in domains II, III, IV (and stable in domain I). In particular, the upper branch of the bistable curve is modulationally unstable. Thus the plane wave hysteresis will not describe the stationary solutions of the system appropriately.

The localized structures we are interested in are calculated numerically from Eqs. (1) assuming rotational symmetry in the (X, Y) plane. We find them coexisting with the stable lower branch of the bistable curve, i.e., they have a plane wave background (see Fig. 2 for an example). The maximum amplitudes (of the fundamental) of localized structures together with the plane wave background

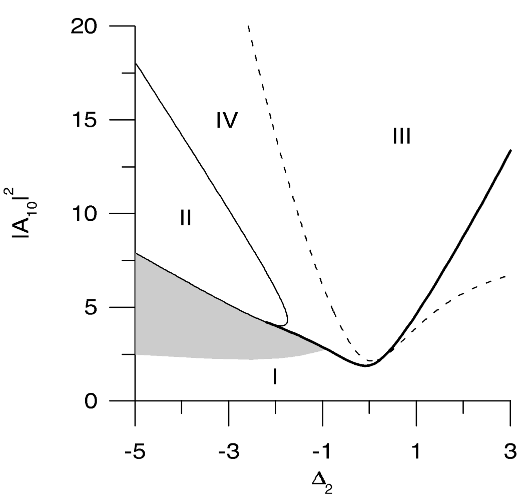

FIG. 1. Loci of critical points in the $(\Delta_2, |A_{10}|^2)$ plane for plane wave solutions ($\Delta_1 = -4$ and $\gamma = 0.5$). The thin solid line marks limit points, the dashed line Hopf bifurcations, and the bold solid line where the modulational instability sets in. In the shadowed part of domain I localized structures exist on a plane-wave background.

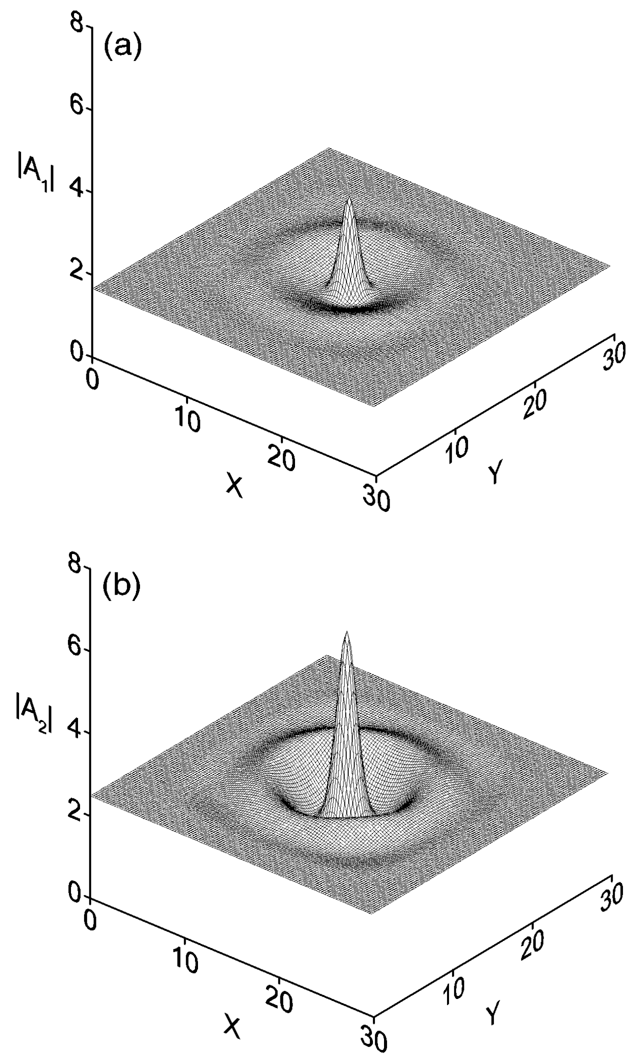

FIG. 2. Amplitude of the (a) fundamental and (b) second harmonics of a localized structure at $\Delta_1 = -4$, $\Delta_2 = -1$, $\gamma = 0.5$, and $E = 4.6$.

are displayed in Fig. 3 for various values of Δ_2 with E as control parameter. The larger the negative detuning Δ_2 the larger the range where they exist. They emanate from the plane wave background with infinitesimal amplitude subcritically, i.e., unstably, and stabilize at a limit point. The stability was checked by means of a two-dimensional beam propagation method. At the point where the localized structures emanate from, the plane waves become modulationally unstable either with finite or infinite period. In the latter case this occurs at a limit point (the same as homogeneously stable). The branch of stable localized structures ends where the background destabilizes, i.e., at the same point in parameter space where they emanate from. From Fig. 3 it is evident that they substitute for the plane wave hysteresis, even outside the range of plane wave bistability or if plane wave bistability does not exist (cf. shadowed part of domain I in Fig. 1 where they exist on a stable plane wave background).

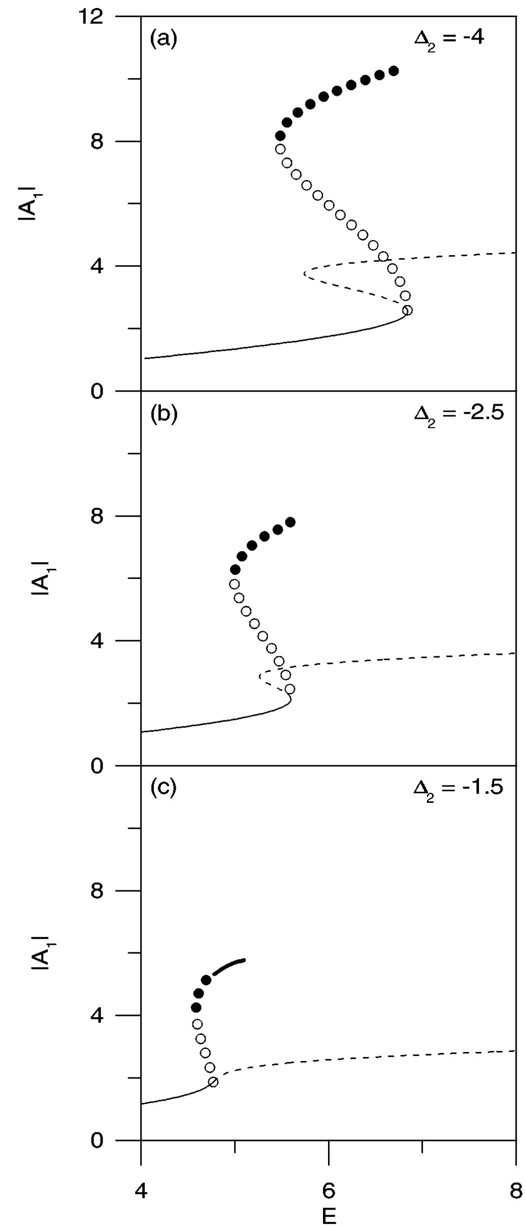

FIG. 3. Maximum amplitudes of the fundamental of localized structures (filled circles: stable; open circles: unstable) and the plane-wave background (solid lines: stable; dashed lines: unstable) in terms of the control parameter E for various values of Δ_2 ($\Delta_1 = -4$ and $\gamma = 0.5$). The bold line in (c) marks maximum amplitudes of stable hexagon patterns.

If there is no plane wave bistability, there is a transition to a hexagon pattern where the stable localized structures terminate in parameter space [bold line in Fig. 3(c)]. Thus they can be considered as residuals of (coexisting) bistable hexagon patterns. This is illustrated in Fig. 4 where the coexistence of a hexagon pattern and the plane wave background is shown. The localized structures may be arranged in an arbitrary way on the plane wave background.

The stable localized structures are strongly localized with very large amplitudes compared to their background. In particular, they generate a large amount of second harmonic field in the center. Thus even a frequency doubling based on stable localized structures could be of some interest for practical applications. In the center of the localized structures their shape corresponds to the one of the solitary waves observed for free space propagation in bulk materials [10]. The resonator influences the tails of the localized structures. The tails are characterized by mainly destructive interference of the localized structures with the plane wave background. In general, they are surrounded by at least one dark ring (cf. Fig. 2). In most cases the creation of a bright localized structure results even in a reduction of the total transmission of the system. Every localized structure acts on the plane wave background like a perturbation. For a stable background which is essential for the existence of stable localized structures this perturbation decays exponentially. There are two cases: they decay in an nonoscillating or oscillating way. If there is

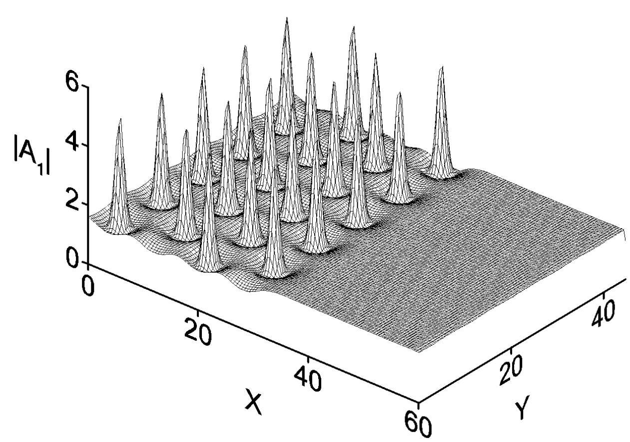

FIG. 4. Amplitude of the fundamental demonstrating the coexistence of a plane wave and a hexagon pattern for $\Delta_2 = -4$, $\Delta_1 = -1.5$, $\gamma = 0.5$, and $E = 4.7$.

no plane wave bistability, there is a transition to the period of the modulational instability approaching the point where it sets in [case of Fig. 3(c)]. This results in many rings, dark and bright (for a not so pronounced example, cf. Fig. 2).

In conclusion, for negative detunings of the fundamental and second harmonics stable localized structures exist on a plane wave background. A collapse as in the case of a cubic nonlinearity does not exist because of the nonlocality of the quadratic interaction. But extreme localization is found. The dissipative nature of the system results in a pronounced interference of the plane wave background with the tails of the localized structures.

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