Vortex Stability and Persistent Currents in Trapped Bose Gases

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(Received 21 May 1997)

We consider the stability of vortices in trapped dilute Bose gases. For weak coupling the vortex is unstable to single-particle excitations: the Bose condensate cannot support persistent currents, and the gas is not superfluid. For stronger coupling the azimuthal symmetry of the rotating condensate is spontaneously broken, resulting in an off-center vortex that eventually spirals out of the trap. [S0031-9007(97)04034-9]

PACS numbers: 03.75.Fi, 05.30.Jp, 32.80.Pj

A dramatic series of experiments with cold, trapped atomic gases has demonstrated Bose-Einstein condensation of a macroscopic number of particles into the same single-particle state [1]. The new condensates are small—millions of atoms confined to clouds tens of microns in diameter—and, unlike liquid helium, are highly compressible. An important question remains: Are these condensates superfluid?

Bulk superfluids are distinguished from normal fluids by their ability to support dissipationless flow [2]. Such "persistent currents" are intimately related to the existence of stable, quantized vortices. Here we consider the stability of azimuthally symmetric quantized vortices in harmonically confined dilute Bose gases. In such rotating condensates, the macroscopically occupied single-particle wave function $\Psi_m(\mathbf{r})$ has azimuthal angular momentum $m\hbar$. We will see that these states are always unstable. When confinement effects dominate atom-atom interactions, the circulating flow decays completely. Although the gas is Bose condensed, it cannot support persistent currents. In the strongly interacting limit we find that confined vortices undergo a spontaneous symmetry breaking that displaces them off axis. For a fixed trap and atomic species, the relative importance of confinement vs interactions can be varied by changing the total number of particles.

Stability.—There are two distinct concepts of stability associated with rotating fluids. First, a rotating state may be produced by steadily driving the fluid with an asymmetric, time-dependent perturbation that rotates at angular velocity Ω about the z axis, relative to the laboratory frame. The steady state of this driven system can be viewed as a state of thermal equilibrium in the corotating frame [3]. The (steady-state) angular momentum in the lab frame then takes the value that globally minimizes $F(L_z, T) - \Omega L_z$, where F is the free energy in the lab frame. If $\partial^2 F/\partial L_z^2$ is positive, this rotating state will be thermodynamically stable *in the continued presence of the driving perturbation.*

The steadily rotating states that are generated by such an externally imposed rotation are not a manifestation of superfluidity—even classical fluids respond to an external rotation by acquiring angular momentum. A fluid is "super" only if the circulating flow persists *in the absence of a rotating drive*. This stability is conceptually distinct from thermodynamic stability, since it concerns a fluid held in a static (rather than rotating) container.

Of course, in the absence of an imposed rotation the global free energy minimum is always a nonrotating state. States with persistent currents are therefore metastable—their (free) energy cannot be reduced by small changes in the state of the system. This "local" stability generally implies the "thermodynamic" stability discussed above, but not vice versa.

For bulk superfluids, the criteria of thermodynamic and local stability coincide. For example, consider a rotating container of liquid ^{4}He. At small angular velocity only the normal component rotates. Above a critical angular velocity, however, the steady state changes discontinuously to a vortex whose superfluid component has quantized circulation [4]. When the rotation of the container is halted, the normal fluid eventually comes to rest, but the circulating superfluid current persists. The vortex is locally stable because few-particle excitations which decrease the angular momentum of the fluid cost free energy. For a large system, this implies macroscopic energy barriers between the metastable vortex and the nonrotating thermodynamic ground state.

We will show that condensation and persistent currents are distinct features of mesoscopic Bose systems. For weakly interacting gases, stable quantized vortices exist only in a driven system, and become unstable when the imposed rotation is halted; the gas rapidly reverts to a nonrotating state. Such Bose condensates cannot be considered superfluid. When interaction and trap effects become comparable, however, we find interesting symmetry-breaking instabilities that modify the vortex core and induce precession about the axis of the trap.

Vortex states.—Consider a collection of N identical bosons of mass M trapped in an axially symmetric harmonic potential $V(r, z)$ at low temperature. We treat interatomic scattering by a delta-function pseudopotential of strength $U = 4\pi\hbar^2 a/M$, where a is the s-wave scattering length. A dimensionless measure of the interactions

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relative to the level spacing of the trap is then

$$\gamma = \frac{UN}{(2\pi)^{3/2}\sigma^3\hbar\omega} = \sqrt{\frac{2}{\pi}}\frac{Na}{\sigma}, \qquad (1)$$

where $\sigma = (\hbar/M\omega)^{1/2}$ is the Gaussian width of the single-particle ground state of the trap. Although we specifically consider a spherically symmetric, harmonic trap, our qualitative results rely only on axial symmetry.

Vortices in trapped gases can be established by driving the system with a weak rotating perturbation. To study the local stability of a vortex, we must first determine the vortex condensate wave function Ψ_m for a range of couplings γ. For the dilute Bose gas, Ψ_m is accurately given by minimizing the total energy per particle [5–7]

$$\frac{E}{N} = \int d^3r\left(\Psi^*\hat{T}\Psi + \hat{V}_{\rm trap}|\Psi|^2 + \frac{UN}{2}|\Psi|^4\right), \quad (2)$$

subject to the constraint that Ψ_m varies as $e^{im\phi}$. (Here $\hat{T} = -\hbar^2\nabla^2/2M$ is the kinetic energy operator.) Dafolvo and Stringari [8] have performed extensive numerical calculations of these states, which are thermodynamically stable for sufficiently large driving angular velocity Ω. Here we briefly recall the limits of small and large γ.

For weak coupling (small γ) the vortex condensate is well approximated by

$$\Psi_m \propto r^m e^{im\phi}\exp\left[-\frac{1}{2\sigma^2}\left(\frac{r^2}{S_r^2} + \frac{z^2}{S_z^2}\right)\right], \qquad (3)$$

where S_r and S_z are dimensionless variational parameters. Equation (3) is exact for $\gamma = 0$, with $S_r = S_z = 1$. In the strong-coupling limit (large γ) the vortex becomes large. The radial and axial kinetic energies are then small, and may be neglected [7]. In this Thomas-Fermi limit Ψ_m is given by [8]

$$|\Psi_m|^2 = \frac{1}{UN}\left(2A - \frac{r^2}{\sigma_r^2} - \frac{z^2}{\sigma_z^2} - \frac{m^2\sigma_r^2}{r^2}\right), \quad (4)$$

where $A(\gamma)$ is chosen to normalize Ψ_m to unit density. The Thomas-Fermi condensate vanishes inside $R_{\rm min}$ and outside $R_{\rm max}$, which are defined by the zeros of Eq. (4). The radius of the trapped cloud is [7] $R_{\rm max} \sim \sigma\gamma^{1/5}$. The core radius $R_{\rm min}$ is given [8] by $m\xi$, where $\xi = (8\pi n_0 a)^{-1/2}$ is the "healing length" of a homogenous Bose gas of density $n_0 \sim \gamma^{2/5}/\sigma^3$, i.e., the peak density of the trapped nonrotating Bose gas. Then $\xi \sim \sigma^2/R_{\rm max}$, so that the core narrows as the cloud grows.

Quasiparticles.—We begin with the weak coupling limit, where the appropriate theoretical framework is provided by the Hartree-Fock approximation. (We will return to the strong coupling limit below.) Quasiparticle states are constructed by transferring one particle from Ψ_m to an orthogonal single-particle state $\psi(r)$. The Schrödinger equation satisfied by ψ is simply

$$\hat{T}\psi + (V + 2UN|\Psi_m|^2 - \mu_m)\psi = \epsilon\psi, \qquad (5)$$

where $\mu_m = \partial E/\partial N$ is the energy gained by removing a particle from the axially symmetric vortex Ψ_m, and ϵ is the quasiparticle excitation energy. Quasiparticle motion is evidently governed by an effective Hartree potential $V_{\rm eff} \equiv V_{\rm trap} + 2UN|\Psi_m|^2 - \mu_m$, which combines attraction to the center of the trap with a mean repulsion by the condensate. The factor of 2 accounts for exchange. When $V_{\rm trap}$ has a minimum near the origin, so does $V_{\rm eff}$ (Fig. 1).

It is easy to see that quasiparticle states at the periphery of the trapped vortex all have positive energy, since $V_{\rm eff}$ is positive there. Core states, however, are localized near the minimum of the trap, and may in principle have negative energy (relative to μ_m). If the core supports such quasiparticle bound states, then collisions with thermally excited particles and asymmetries of the trap will drive particles from the condensate to the core state, destabilizing the vortex. Since the quasiparticles of a Bose gas are themselves bosons, the core state can become macroscopically occupied [9].

Core states.—It is easy to see that the core quasiparticle state *always* has negative energy relative to the vortex condensate for $\gamma = 0$, since then the "core" state of (5) is nothing but the ground state of the trap, while Ψ_m is the lowest energy state of angular momentum $m\hbar$. Treating interactions perturbatively we find that the net energy cost to transfer a particle from the vortex to the core is then $\epsilon = -m\hbar\omega_r + O(\gamma)$, which is always negative for small γ. The undriven vortex Ψ_m is therefore unstable. This conclusion is independent of the details of the trap potential.

As γ increases, however, the core of an $m = 1$ vortex becomes very narrow, and the zero-point motion of a particle confined to the cylindrical core exceeds μ_m. The Schrödinger equation (5) then has no bound state, and Ψ_m becomes stable with respect to single-particle excitations. (Using a simple Gaussian variational calculation, the bound state is lost for $\gamma \gtrsim 5$.) The strong coupling limit will be discussed further below.

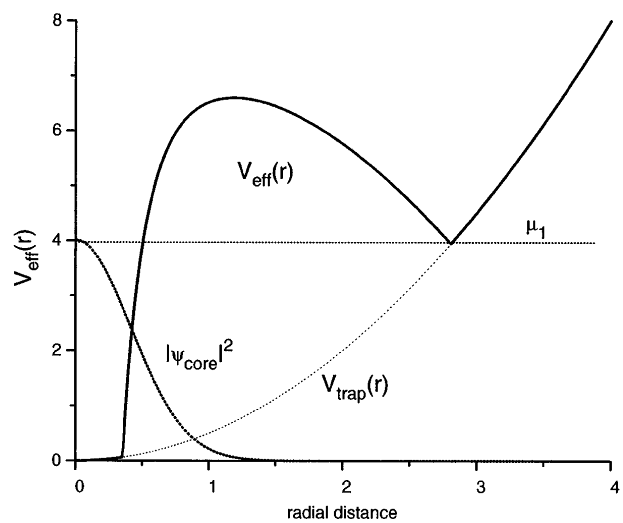

FIG. 1. The effective potential $V_{\rm trap} + 2UN|\Psi_m|^2$ is shown as a function of radial distance for $z = 0$, for $\gamma = 5$. Also shown is a Gaussian variational core state.

For $m \geq 2$ the centrifugal forces on the condensate are larger than for the unit vortex, and the core radius $R_{\rm min}$ expands by roughly a factor of m. The kinetic energy of the core state therefore decreases by a factor of $1/m^2$. It is easy to see variationally that for $m \geq 2$ there are bound core states for *all* γ. An azimuthally symmetric vortex with $m \geq 2$ is therefore *always* locally unstable.

Instability.— What is the fate of an unstable vortex? If we prepare the Bose gas in an axially symmetric rotating state and then suddenly turn off the driving perturbation, the gas is not in its lowest free energy state, and responds by transferring particles from the vortex condensate to the core [10]. For small core-state occupation $N_{\rm core}$, we may assume that Ψ_m and $\psi_{\rm core}$ are unchanged by this transfer. The total energy is then approximately

$$E[N_{\rm core}] = N_{\rm core}\epsilon + \frac{UN_{\rm core}^2}{2}\int d^3r|\psi_{\rm core}|^4, \quad (6)$$

where we have neglected the contribution of the normal gas at low temperature, and the last term accounts for quasiparticle-quasiparticle repulsion in the core. Equation (6) is minimized for macroscopic core occupation:

$$x_{\rm core} \equiv \frac{N_{\rm core}}{N} \approx \frac{|\epsilon|s_r^2 s_z}{\gamma\hbar\omega}, \quad (7)$$

where s_r and s_z are the radial and axial widths of the bound state. When the core-state volume $s_r^2 s_z \sim \xi^2 R_{\rm max} \sim \gamma^{-1/5}$ is small, $x_{\rm core}$ is small. (For $\gamma = 3$, e.g., $x_{\rm core}$ is less than 10%.) The result is a vortex that coexists with a smaller, nonrotating condensate trapped in its core. The properties of this unusual double Bose condensate are discussed below.

For small γ, Eq. (7) predicts $N_{\rm core} \sim N$, and the assumption that Ψ_m and $\psi_{\rm core}$ are unchanged by occupation of the core state breaks down. They must instead be determined self-consistently, including the force that the core particles exert on those left behind in Ψ_m. These effects broaden the core, further destabilizing the vortex condensate. Ultimately, all the particles will be transferred to the $m = 0$ core state, resulting in a nonrotating Bose condensate. We can estimate the critical coupling for complete destruction of the vortex by the γ at which $x_{\rm core} \sim 1/2$ in Eq. (7). For $m = 1$ we find $\gamma_{c1} \sim 1.3$; for $m = 2$, $\gamma_{c2} \sim 1.8$.

Symmetry breaking.—We have argued that above a critical coupling γ, a rotating, trapped Bose gas can exist in a novel state with coexisting condensates of differing angular momenta—a vortex condensate Ψ_m and a nonrotating core condensate $\psi_{\rm core}$. In a perfectly symmetric potential, Josephson tunneling between the two condensates is forbidden by angular momentum conservation, and their phases are not coupled. Any weak asymmetry of the trap, however, will initiate coherent Josephson scattering between the two condensates, inducing phase coherence between them. The resulting many-body ground state is described by a macroscopically occupied state whose condensate wave function is the superposition [11]

$$\Psi = \sqrt{1 - x_{\rm core}}\,\Psi_m + e^{i\chi}\sqrt{x_{\rm core}}\,\psi_{\rm core}. \quad (8)$$

The relative phase χ is discussed below. The induced coherence between the two condensates is a finite response to an infinitesimal perturbation, and signals the spontaneous breaking of rotational symmetry, as we see next.

What kind of state is Eq. (8)? Consider the $m = 1$ vortex plus its core. Without loss of generality we take $\chi = 0$. The core state is real and positive everywhere, peaked near the z axis. The real part of $\Psi_{m=1}$, however, is positive for $x > 0$ and negative for $x < 0$. Superimposing these states [Fig. 2(a)], we see that the node of the $m = 1$ vortex is displaced away the trap axis. The magnitude of the displacement is comparable to $R_{\rm min}$.

The displacement of the core has a simple physical explanation. In the axially symmetric vortex Ψ_m, the node passes through the minimum of the trap potential. By displacing the vortex core, the particle density in this energetically desirable location is increased. The resulting density $N|\Psi|^2$ is asymmetrical, and the center-of-mass of the gas is not at the origin. (The on-center state can be stable only if the vortex core is "pinned" to the center of the trap [4] by a locally repulsive trap potential [12].)

By Ehrenfest's theorem, it follows that a displaced vortex will precess about the axis of the trap at the bare trap frequency ω, so that $\chi(t) = \chi_0 + \omega t$. Above γ_{c1} in a harmonic trap, we therefore expect to find a precessing, off-center vortex. At nonzero temperature, this precessing vortex moves through normal fluid, and the resulting dissipation [13] will cause the vortex to spiral outward.

A similar analysis can be carried out for the $m = 2$ vortex, which is unstable for all γ. Again, the core state $\psi_{\rm core}$ is positive and peaked near the origin, while $\Psi_{m=2}$ varies as $e^{2i\phi}$. For $\chi = 0$, these two states constructively interfere along the x axis, and destructively interfere along the y axis [Fig. 2(b)]. The new condensate (8) then has *two* nodes, symmetrically placed on either side of the origin along the y axis at a distance comparable to the naive core size. Again, they precess at frequency ω. Similar reasoning applied to vortices with $m > 2$ yields a ring of equally spaced unit [14] vortices precessing about the center of the trap. The splitting of $m \geq 2$ vortex cores is the mesoscopic quantum analog of the hydrodynamic instability [13,15] of high angular momentum vortices towards splitting into multiple "fundamental" vortices with $m = 1$.

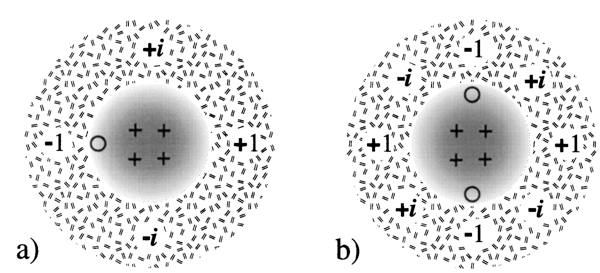

FIG. 2. Schematic of Ψ_m (stippled) and $\psi_{\rm core}$ (shaded) for $m = 1$ (a) and $m = 2$ (b) in the XY plane. Combining Ψ_m and $\psi_{\rm core}$ yields new states with nodes indicated by small open circles.

Strong coupling.—We may consider the stability of an axially symmetric vortex by another means. If Ψ_m is a local minimum of the Gross-Pitaevskii energy functional Eq. (2), then small variations of the form $\Psi = \Psi_m + \psi$ must increase its value. It is easy to show, however, that

$$\delta E = \int d^3 r \psi^* [\hat{T} + V_{\mathrm{eff}}]\psi + (UN/2) \times [\Psi_m^2 (\psi^*)^2 + (\Psi_m^*)^2 \psi^2] \quad (9)$$

can be negative. In particular, the fact that $\hat{T} + \hat{V}_{\mathrm{eff}}$ has a (real) bound state implies that δE is negative for the corresponding variation ψ_{core}. The axially symmetric state Ψ_m is therefore *not* a local minimum of Eq. (2), and we see again that the on-axis vortex is unstable. [The Schrödinger equation obtained by extremizing the quadratic form (9) is distinct from, but closely related to, the Bogoliubov equations [12].]

Armed with this new interpretation of the instability, we return to the strong coupling limit. The vortex condensate density $|\Psi_m|^2$ then has three distinct regimes: a core region $r \lesssim \xi$, a broad plateau near $r \sim \sigma$, and a gradual decline to the edge of the cloud at R_{max}. From Pitaevskii's original calculation [6] we know that the spectrum of a vortex in a *homogeneous* Bose gas contains a zero-energy mode, a result which follows from translational invariance. The resemblance between the core and plateau regions of the trapped and homogeneous vortices suggests that we use Pitaevskii's zero mode of the homogeneous system as a variational state in (9) for $r \lesssim \sigma$, with a suitably smooth spatial cutoff near $r \gtrsim \sigma$. It is then not difficult to show [12] that (9) can be made negative for all γ. Evidently, the instability is a displacement of the core of the vortex, leaving the far-field condensate unchanged [16]. Again, Ehrenfest's theorem implies precession, which is accompanied by dissipation.

Our analysis is supported by a reinterpretation of the Bogoliubov spectrum of an axially symmetric vortex obtained by Dodd *et al.* [17]. In the weak coupling limit ($N \approx 0$) they find *four* excitations with energy $+\hbar\omega_r$ above the $m = 1$ vortex. From the spectrum of a single particle in a harmonic potential, however, we know that there are only *three* such excited states for $\gamma = 0$. A fourth "excited" state is found at *negative* energy $-\hbar\omega_r$ (i.e., our core state). It follows that the excitation (with $m = 0, 2$) found by Dodd *et al.* is in fact the Bogoliubov *antiparticle* to our core mode [12]. Since this antiparticle energy is positive for all γ, the core mode itself always has negative energy. We conclude that vortices in harmonic traps are *never* locally stable.

In conclusion, by considering the stability of vortex states, we have argued that Bose condensates in harmonically trapped Bose gases do not support persistent currents. Although rotating, quantized condensates can be produced in a steadily rotating trap, when the imposed rotation ceases the vortex either decays by a single-particle instability or precesses about the trap axis. At nonzero temperature the precessive motion of the vortex core through the normal fluid dissipates energy and angular momentum, which eventually brings the fluid to rest.

I thank D. Weiss, J.C. Davis, E. Cornell, W. Ketterle, R. Packard, and D.P. Arovas for exciting discussions. This work was supported by NSF PYI-91-57414, the UCB Committee on Research, and the Miller Institute for Basic Research.

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