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Quantum Theory as the Representation Theory of Symmetries

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It is proposed that an unconstrained quantum system is completely specified by its configuration space C and its group of symmetries G. It is then shown that the theory of projective unitary representations of G on a certain Hilbert space determined by C and G leads to a definitive resolution of supposed ambiguities relating to superselection rules, choice of wave functions, topological properties of G and C, anomalies, etc. [S0031-9007(97)04114-8]

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The word quantization generally signifies a procedure which starts with a classical description of a dynamical system and seeks its exact quantum generalization. In its most practical form, Schrödinger quantization, it consists in taking the state space $\mathcal{H}$ to be a Hilbert space of square-integrable complex functions on the configuration manifold C and the observables to be self-adjoint operators on $\mathcal{H}$ determined, by some prescription, from real functions of the local coordinates of C and their time derivatives (functions on the tangent bundle of C). From time to time, this procedure has been found not to result in a sufficiently detailed or accurate description of quantum phenomena, or to reflect certain perceived subtleties of the classical description, and *ad hoc* modifications prescribed accordingly. Examples which come to mind include the occurrence and origin of superselection (SS) rules, the realization of SS sectors as spaces of (multivalued) wave functions, the fixing of boundary conditions, the interpretation and treatment of anomalies, the role of the topology of C, and the use of the universal covering groups (equivalently, the Lie algebras) of Lie groups of symmetries, etc. These or similar ambiguities affect other quantization procedures—e.g., canonical quantization (and its modern rigorous version, geometric quantization) and path-integral methods—as well.

Logically, it should cause no surprise that any approach to defining the fundamental principles governing the exact (quantum) reality starting from an approximate (classical) description might require occasional tinkering. Equally, such a procedure is necessarily of tentative validity, to be replaced ultimately by a formulation using exclusively concepts and methods appropriate to quantum physics. Such an autonomous formulation of the general principles of quantum theory is put forward and justified in this Letter.

There already exists a full fledged program of a strictly quantum formulation of quantum theory of great generality: the algebraic quantum theory based on an axiomatization of the notion of observables [1] due to Haag and his collaborators. The present proposal is less ambitious in its scope and very different in technical detail. The key role here is played by the symmetry group G of a given system (G forms part of the "data" defining the system). How to proceed from the knowledge of G to a complete working out of quantum kinematics and dynamics is first concisely described below in an inductive manner and finally formulated as a precise postulate. This order of presentation serves well as a guide to the physical motivation as well as to show that, technically, little needs to be added to the existing body of the quantum theory of symmetry. The theoretical framework that emerges, being very concrete and system specific, will be seen to provide clear answers to the sort of problems raised in the introductory paragraph. Though it does not encompass the very broad horizons of the theory of observables, it is general enough to handle any properly specified system. The present account, however, confines itself to unconstrained nonrelativistic systems, postponing a treatment of the generalization required to deal with relativistic quantum fields, including gauge fields and the concomitant constraints, to a sequel.

The foundations of the theory of quantum symmetries were laid in the work of Weyl [2], Wigner [3], and Bargmann [4]. This will be our starting point. Given that a state of a system is a ray of some complex Hilbert space $\mathcal{H}$ (as follows from the superposition principle [5]), a symmetry of the system is a one-to-one map of the projective space $P\mathcal{H}$ into itself fixing absolute values of scalar products in $\mathcal{H}$ [3]. From Wigner's theorem, any group of symmetries is represented in $\mathcal{H}$ by a projective unitary representation (PUR). (We disregard circumstances leading to antiunitary representations.) The first task is thus to classify the PURs of the group $G = \{g, h, \dots\}$ of all symmetries of the system, called simply the symmetry group from now on.

A PUR U of G satisfies $U(g)\,U(h) = \gamma(g,h)\,U(gh)$, where γ is a U(1)-valued function on $G \times G$ subject to the conditions $\gamma(gh,k)\,\gamma(g,h) = \gamma(g,hk)\,\gamma(h,k)$ and $\gamma(g,1) = \gamma(1,g) = 1$, in other words, a 2-cocycle. If γ is such that $\gamma(g,h) = \beta(g)\beta(h)\beta(gh)^{-1}$ for some function β on G, it is a coboundary and the corresponding PUR is unitary equivalent to a UR. Hence PURs fall into equivalence classes labeled by the elements of the quotient of the Abelian group of 2-cocycles by its subgroup of coboundaries, $Z^2(G,\mathrm{U}(1))/B^2(G,\mathrm{U}(1)) = H^2(G,\mathrm{U}(1))$, the second cohomology group of G with coefficients in U(1). $H^2(G,\mathrm{U}(1))$ also classifies (equivalence classes of) central extensions of G by U(1), i.e., groups $\tilde{G}$ having U(1) as a central subgroup such that $\tilde{G}/\mathrm{U}(1) = G$. The direct product $G \times \mathrm{U}(1)$ is the trivial central extension and is the only extension having G as a subgroup. An extension $\tilde{G}_\eta$ corresponding to $\eta \in H^2(G,\mathrm{U}(1))$ can be given concretely as the group of pairs (g,a) for the group law $(g,a)\,(h,b) = (gh, \gamma(g,h)ab)$, a, $b \in \mathrm{U}(1)$, for any representative cocycle γ in the class of η. It follows that if $\tilde{U}$ is a UR of $\tilde{G}_\eta$ such that its restriction to the U(1) subgroup is the natural character $\tilde{U}(a) = a$, $\tilde{U}$ satisfies $\tilde{U}(g,1)\,\tilde{U}(h,1) = \gamma(g,h)\,\tilde{U}(gh,1)$; i.e., the restriction of $\tilde{U}$ to the subset of G of $\tilde{G}_\eta$, $\tilde{U}(g,1) \equiv U(g)$, gives a PUR corresponding to η. The converse is also true: Every PUR of G lifts to a UR of some central extension of G by U(1) with the property that it restricts to the central U(1) as the natural UR. Thus the determination of all PURs of G is reduced to the determination of URs, restricting to U(1) naturally (this property is henceforth understood), of $\tilde{G}_\eta$ for every η. In particular, if $H^2(G,\mathrm{U}(1)) = 0$, G has only trivial extensions and all its PURs are URs (or trivial PURs).

From these well-known facts (for a systematic recent account, see [6]), significant conclusions can already be drawn. If $\mathcal{H}_\eta^1$ and $\mathcal{H}_\eta^2$ for a fixed η carry URs of $\tilde{G}_\eta$, then obviously so does $\mathcal{H}_\eta^1 \oplus \mathcal{H}_\eta^2$; if every vector in $\mathcal{H}_\eta^1$ and $\mathcal{H}_\eta^2$ represent states of a system with symmetry group G, the superposition principle holds without restriction in $\mathcal{H}_\eta^1 \oplus \mathcal{H}_\eta^2$. But for $\eta_1 \neq \eta_2$, the linear superposition $c_{\eta_1}\psi_{\eta_1} + c_{\eta_2}\psi_{\eta_2} \in \mathcal{H}_{\eta_1} \oplus \mathcal{H}_{\eta_2}$ transforms projective unitarily under G only if $c_{\eta_1} = 0$ or $c_{\eta_2} = 0$. (This is almost evident; for the elementary proof see [6].) Such a vector cannot represent a state since G must act projective unitarily on all states. A superselection rule separates the sectors $\{P\mathcal{H}_\eta\}$, and there is no linear space containing *all* states as rays and *all* of whose rays are states.

There is a powerful general method of dealing simultaneously with all SS sectors, *via* the notion of the universal central extension $\hat{G}$ of G. $\hat{G}$ is a group from which there is a homomorphism $\varphi_\eta \colon \hat{G} \to \tilde{G}_\eta$ for every η such that a UR of $\tilde{G}_\eta$, on composing with φ_η, lifts to a UR of $\hat{G}$; so every sector of G is in some UR of $\hat{G}$ [7]. For groups of interest to us, $\hat{G}$ is a unique nontrivial central extension of G by the group $\Sigma(G) = H^2(G,\mathrm{U}(1))^*$, the Pontryagin dual (the group of characters) of $H^2(G,\mathrm{U}(1))$. (For the general theory, see [7].)

A UR $\hat{\mathcal{H}}$ of $\hat{G}$ can be decomposed as a direct sum of subspaces $\{\mathcal{H}_\eta\}$ corresponding to the characters η of $\Sigma(G)$; η is indeed an element of $H^2(G,\mathrm{U}(1))$ since, by Pontryagin duality, $\Sigma(G)^* = H^2(G,\mathrm{U}(1))$. In other words, $\Sigma(G)$ decomposes $\hat{\mathcal{H}}$ into SS sectors and the state space is a family $\{P\mathcal{H}_\eta\}$, exactly as required.

The (Abelian) "superselection group" $\Sigma(G)$ plays a primordial role in all our considerations.

Since $\Sigma(G)$ acts on all of $\mathcal{H}_\eta$ by the same character, it commutes with all operators mapping $\mathcal{H}_\eta$ into itself. Defining an observable to be any self-adjoint operator mapping states into states as required by the axiomatic demand [5] that an observation causes a transition to a state, it must necessarily map each sector into itself (otherwise its eigenvectors will span more than one sector and cannot represent states) and so commute with $\Sigma(G)$. Thus the SS structure, determined solely by G, decides which operators are observables. The direction of implications here is the reverse of that in algebraic quantum theory in which, in a wider context, SS sectors are obtained as certain special inequivalent representations of the observable algebra [8]. (For the current status and for a complete bibliography of the subject, see [9].)

Which specific URs of $\{\tilde{G}_\eta\}$—the irreducible URs and their multiplicities—constitute the sectors of the state space of a system? Equivalently, which UR of $\hat{G}$ is to be chosen as $\hat{\mathcal{H}}$? The answer suggested by the success of the simplest version of Schrödinger quantization is the space of wave functions. We are led to assume therefore that the system is specified by, besides G, a second datum, the configuration space C, and that G acts on C as a transformation group. (C, as the set of all "positions" of a system, is a legitimate concept in the quantum context.) Suppose to begin with that $H^2(G,\mathrm{U}(1)) = 0$. Schrödinger quantization then identifies the state space $\mathcal{H}_0$ with $L^2(C)$, the space of complex functions ψ on C square integrable with respect to a G-invariant measure. G has a UR on $L^2(C)$ given by $(U(g)\psi)(x) = \psi(g^{-1} \cdot x)$. In this generality (accommodating the possibility of G, or both G and C, being discrete), the decomposition of $L^2(C)$

into irreducibles may not always be easy though fully determined in principle. An especially easy situation occurs when G is a compact Lie group and C is the manifold of G on which G acts by left translation. $\mathcal{H}_0$ is then the regular representation $L^2(G)$ containing every irreducible UR with multiplicity equal to dimension (Peter-Weyl theorem).

For nontrivial sectors this procedure is inadequate. In general, the only action of $\tilde{G}_\eta$ on C that will pass to the given action of the quotient group G is the trivially extended one in which U(1) fixes all points of C, $ax = x$ implying $\tilde{U}(1,a)\psi = \psi$. This contradicts the requirement of natural central character, $\tilde{U}(1,a)\psi = a\psi$; so wave "functions" cannot be functions on C. A purely representation-theoretic way of finding the necessary generalization proceeds as follows. Assume first that $C = G$ and consider the regular representation of $\hat{G}$, $L^2(G \times \Sigma(G)) = \hat{\mathcal{H}}$, on which $\hat{G}$ acts unitarily by $(\hat{U}(x,\chi)^{-1}\hat{\psi})(x',\chi') = \hat{\psi}((x,\chi)\cdot(x',\chi')) = \hat{\psi}((x\cdot x', \omega(x,x')\chi\chi')$ where ω is the $\Sigma(G)$-valued 2-cocycle defining $\hat{G}$, $x, x' \in G = C$, and $\chi, \chi' \in \Sigma(G)$. Decomposing $L^2(G \times \Sigma(G))$ by the characters of $\Sigma(G)$, we have $\hat{\mathcal{H}} = \oplus_\eta \mathcal{H}_\eta$ with every η occurring in the sum. By the construction of $\hat{G}$, each summand $\mathcal{H}_\eta$ is a UR of $\tilde{G}_\eta$. On a wave function $\hat{\psi}_\eta \in \mathcal{H}_\eta$, we have $\eta(\chi')\hat{\psi}_\eta(x,\chi) = (\hat{U}(1,\chi')\hat{\psi}_\eta)(x,\chi) = \hat{\psi}_\eta(x,\chi\chi'^{-1})$ using the fact that $\omega(1,x) = 1$. We may then define a function $\psi_\eta \in L^2(G)$ by $\hat{\psi}_\eta(x,\chi^{-1}) = \eta(\chi)\hat{\psi}_\eta(x,1) \equiv \eta(\chi)\psi_\eta(x)$. In particular, $\mathcal{H}_0 = L^2(G)$.

Even when $C \neq G$, the above construction can be shown to be essentially correct: There exist functions ω: $G \times C \to \Sigma(G)$ satisfying $\omega(1,x) = 1$ such that $\hat{G}$ is unitarily represented in $\hat{\mathcal{H}} = L^2(C \times \Sigma(G))$ by $(\hat{U}(g,\chi)^{-1}\hat{\psi})(x,\chi') = \hat{\psi}(g\cdot x, \omega(g,x)\chi\chi')$. Thus the wave functions in $\mathcal{H}_\eta$ form the subspace of the space of "sections of a $\Sigma(G)$-bundle over C" transforming by the character η of $\Sigma(G)$. [The geometric nomenclature does not exclude the possibility of C, G, or $\Sigma(G)$ being discrete.] If one insists on thinking of $\hat{\psi}$ as a function of C, it is necessarily multivalued except for $\eta = 0$.

When G is indeed a (connected) Lie group, these general considerations can be reexpressed in more familiar terms. First, it is possible to define $H^2(G,\mathrm{U}(1))$ in a way consistent with the algebraic theory so far invoked such that $\tilde{G}_\eta$ for every η and $\hat{G}$ are themselves Lie groups to which every continuous PUR of G lifts as a continuous UR (for a simple and elegant account, see [7]). If G is also simply connected, $H^2(G,\mathrm{U}(1))$ is canonically isomorphic to the Lie algebra cohomology $H^2(\mathrm{Lie}\,G,\mathbf{R})$ [4,7,10]. The SS structure is then of strictly algebraic origin, independent of the topology of C. On the other hand, for a semisimple G, $H^2(\mathrm{Lie}\,G,\mathbf{R}) = 0$ while $H^2(G,\mathrm{U}(1)) = \pi_1(G)^*$ [7]; so, by duality, $\Sigma(G) = \pi_1(G)$ and $\hat{G}$ is the universal covering group $\overline{G}$. The SS structure then comes solely from the topology of G. (The topology of C is always irrelevant.) It is therefore permissible to replace G by $\overline{G}$ (or Lie G) only if G is semisimple and, even then, only on imposing the SS structure and all its consequences. For instance, the rotation group is SO(3), not $\mathrm{SU}(2) = \hat{SO}(3)$, and a rotation-invariant system will have URs of SU(2) in $\hat{\mathcal{H}}$, but subject to the action of the univalence SS group $\Sigma(\mathrm{SO}(3)) = \pi_1(\mathrm{SO}(3)) = \mathbf{Z}_2$. A nonsemisimple example is SO(2): $\pi_1(\mathrm{SO}(2)) = \mathbf{Z}$ has nothing to do with $\Sigma(\mathrm{SO}(2))$, which vanishes [4,6]. SO(2) has no PURs other than URs, implying that angular momentum in $\mathbf{R}^2$ is integral and ruling out the existence of anyons (defined as particle states having arbitrary real spins [11,12]). However, since the 2 + 1 Lorentz group $L(2,1)$ is semisimple, with $H^2(L(2,1),\mathrm{U}(1)) = \pi_1(L(2,1))^* = \mathrm{U}(1) = \{0 \leq \theta < 2\pi\}$, there do exist PURs of the 2 + 1 Poincaré group having any real "Lorentz helicity" [6]. The corresponding free fields may legitimately be termed anyonic fields and the parameter θ identified with an Abelian character of the braid group B_n for arbitrary n just as, in 3 + 1 dimensions, the spin-statistics connection identifies $H^2(L(3,1),\mathrm{U}(1))^* = \mathbf{Z}_2$ with the Abelianization of the permutation group, $S_n/A_n = \mathbf{Z}_2$. (For a study of PURs of the 2 + 1 Galilei group and how they embed URs of SO(2), see [13]; for the reconciliation of the existence of anyonic fields with the absence of anyonic particles, see [6].)

We turn now to the correct formulation of dynamics in nontrivial sectors, in brief résumé ([6] has an extensive treatment). It is most easily done, once again, for $C = G$, with G a Lie group, assumed simply connected so that the SS structure is purely algebraic. Then $\{X_i\}$, a basis for Lie G, has the dual significance of being both the conserved charges and the generalized velocities. In the trivial PUR U_0 of G, the kinetic energy is then an operator $T_0 = d_{ij}\; U_0(X_i)\; U_0(X_j)$ where d is a real symmetric (so that T_0 is self-adjoint) nondegenerate (there being no constraints) matrix. We may write T_0 simply as the element $d_{ij}X_iX_j$ of the universal enveloping algebra $\mathcal{U}(G)$ of Lie G and simultaneously identify it—since, by the symmetry of d, X_i and X_j commute inside T_0—with the classical kinetic energy. The matrix d must in addition be such that $[T_0, X_i] = 0$. By virtue of the Heisenberg equation of motion, which is no more than the definition of the Hamiltonian as the generator of time translations (for the effect of adding a potential energy, see below), this condition expresses simultaneously the G invariance of T_0 and the conservation of the charges $\{X_i\}$.

For $\eta \neq 0$, $\{X_i\}$ form only a vector space basis of Lie $\tilde{G}_\eta$. Nevertheless, symmetric polynomials, in particular, T_0, belong to $\mathcal{U}(\tilde{G}_\eta)$ and can be algebraically manipulated in $\mathcal{U}(\tilde{G}_\eta)$ by defining the commutator $[X_i,X_j]_\eta = [X_i,X_j]_0 + \Gamma^\eta_{ij}$ where $[\,,\,]_0$ is the Lie bracket of Lie G and Γ^η_{ij}, and the central charges are the values of the Lie algebra 2-cocycle Γ^η on pairs (X_i,X_j). An easy computation gives $[T_0, X]_\eta = -2\Gamma^\eta dX \equiv \alpha_\eta X$ for any $X \in \text{Lie } G$. The "anomaly" α_η is a linear operator on Lie G as a vector space which is zero if and only if Γ^η is a coboundary. Thus the charges appear not to be conserved if time derivatives are computed in $\mathcal{H}_\eta$, $\eta \neq 0$, using the G-invariant Hamiltonian of $\mathcal{H}_0$. More pertinently, T_0 is not $\tilde{G}_\eta$ invariant—the anomaly reflects a mismatch of sectors. The correct Hamiltonian is given by $T_\eta = T_0 + t_\eta$, where t_η is an element of Lie G solving the anomaly equation $[t_\eta, X] = \alpha_\eta X$, which just expresses the $\tilde{G}_\eta$ invariance of T_η. (See [6] for details and the example below for illustration.)

To round off, here are some miscellaneous remarks on dynamics. Adding a potential energy V only requires G to be replaced by its subgroup leaving V invariant. The restriction $C = G$ can also be quite easily removed. (Topological SS structure requires more work.) Discrete systems are amenable to the same general procedure, keeping in mind that the Hamiltonian is then a (nonunique) element of the group algebra of G [14].

With the evidence in hand, we are in a position to state the general postulate:

A constraint-free quantum system is specified by giving a configuration space C and a group of symmetries G acting on C as a transformation group. Its state space is the family of projective Hilbert spaces $\{P\mathcal{H}_\eta, \eta \in H^2(G, U(1))\}$ where $\mathcal{H}_\eta$ is the subspace of $\hat{\mathcal{H}} = L^2(C \times H^2(G, U(1))^*)$ on which the UR of the universal central extension $\hat{G}$ of G carried by it restricts as the character η of $H^2(G, U(1))^* \subset center\ \hat{G}$.

All the phenomena treated in the main part of the paper follow from this postulate. It is unambiguous, expressed in terms of well-defined physical concepts (namely, G, and C) [15], and the consequences can be explored with clarity and precision.

As an example, consider the Euclidean-invariant motion of a particle of mass $= \frac{1}{2}$ in the plane: $C = \mathbf{R}^2$, $G = E(2) \equiv \text{SO}(2) \vec{\times} \mathbf{R}^2$, the semidirect product of the rotation and translation groups. We have [4,6] $H^2(E(2), \text{U}(1)) = H^2(\mathbf{R}^2, \text{U}(1)) = \mathbf{R}$; equivalence classes of PURs of $E(2)$, labeled by a real number η, are in one-to-one correspondence with those of $\mathbf{R}^2$ and the central extensions are $\tilde{E}(2)_\eta = \text{SO}(2) \vec{\times} \tilde{\mathbf{R}}^2_\eta$. The trivial sector accommodates the quantum mechanics of the free particle. The SS structure is purely algebraic. For $\eta \neq 0$, the Heisenberg group $\tilde{\mathbf{R}}^2_\eta$ is obtained by exponentiating its Lie algebra $[P_i, P_j] = i\eta\varepsilon_{ij}$ of momenta (the generators of space translations) and has a unique (up to equivalence) irreducible UR. It can then be shown [14] that irreducible URs of $\tilde{E}(2)_\eta$ form a set $\{\mathcal{H}_{\eta,j}, j \in \mathbf{Z}\}$ in which $\mathcal{H}_{\eta,j}$ for every j is the irreducible UR of $\tilde{\mathbf{R}}^2_\eta$ up to a j-dependent equivalence where j is the maximum angular momentum in $\mathcal{H}_{\eta,j}$. For each η, the η sector is $\mathcal{H}_\eta = L^2(\mathbf{R}^2)$, having the decomposition $\mathcal{H}_\eta = \oplus_j \mathcal{H}_{\eta,j}$. The $E(2)$-invariant free Hamiltonian $T_0 = P_i P_i$ is, of course, not $\tilde{E}(2)_\eta$ invariant for $\eta \neq 0$ and the anomaly equation has the solution $t_\eta = 2\eta J \in \text{Lie } E(2)$ (J is the angular momentum), giving $T_\eta = P_i P_i + 2\eta J$. The resulting equation of motion is then the Lorentz force equation for a particle of charge e in a uniform magnetic field $B = \eta/e$. In fact, the entire Landau theory follows just from the symmetry (and without having to invoke a vector potential) [14,16]. The method can be readily adapted to deal with discrete groups (periodic potentials) and/or discrete configuration spaces (lattices), $C = G = \mathbf{Z}^2$ [14], $C = \mathbf{R}^2$, $G = \mathbf{Z}^2$ [16].

In conclusion, the relationship between constant magnetic filed problems and PURs of translation groups has a natural generalization covering Poincaré-invariant quantum U(1)-gauge field theory. The key to this lies in the localizability property of certain (nontachyonic) PURs of $P(3,1)$ [17], taking account of which requires a generalization of the cohomological machinery used here. This as well as a more general consideration of relativistic symmetries will be taken up elsewhere.

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observation on "spin anyons" is meant mainly to highlight the physical consequences of the distinction between the universal central extension and the universal cover.

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