Measurements of Radiation Pressure Effect in Cryogenic Sapphire Dielectric Resonators

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We report observations of the radiation pressure induced expansion of a solid dielectric resonator. This effect causes a fundamental limit to the frequency stability of the resonator. The measurements were made on four high Q-factor quasi-TE "whispering gallery" modes from 9.9 to 12.6 GHz in a monocrystalline sapphire resonator at liquid helium temperatures. The fractional frequency shift is $-1.0 \pm 0.1 \times 10^{-12}$ per μJ of energy stored in the resonator. This result is consistent in sign, magnitude, and linearity with the radiation pressure induced lattice expansion term predicted by Braginsky. [S0031-9007(97)03870-2]

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Fundamental limits to the stability of macroscopic electromagnetic oscillators arise according to the uncertainty principle, from the competing effects of amplifier noise and radiation pressure. As the signal power increases, quantum fluctuations in the radiation density within the resonator leads to a fluctuating radiation pressure which directly modulates the frequency of the dielectric resonator. This phenomenon was first considered by Braginsky *et al.* [1]. Today the radiation pressure fluctuation effects are still beyond the scope of experiments. Here we present data which confirm the static radiation pressure deformation predicted by Braginsky, which itself confirms the existence of a quantum limit to frequency stability which is 5 orders of magnitude smaller than presently achievable sapphire oscillator performance.

At the University of Western Australia, we have been undertaking experimental investigations into high stability cryogenic sapphire resonators for secondary frequency standards. The resonator is based around a 5 cm diameter cylindrical monocrystalline sapphire element supported by monolithic spindles inside a niobium cavity [2–4]. The resonator's mode frequency dependence on dissipated power or stored energy is important in high stability applications as it will determine the required level of power control to achieve a particular frequency stability. In this paper we report observations of the radiation pressure deformation of a solid dielectric resonator crystal using four resonator modes of the same family. Preliminary measurements of the power dependence of only one mode have been reported by us previously over a much smaller power range [5]. Other measurements of resonator power dependence have been dominated by the power dependence of the surface reactance of the nearby superconducting walls to the extent that the radiation pressure term is not visible [6,7]. Braginsky *et al.* measured the power dependence in a cryogenic sapphire resonator coated with a niobium film [6] and could not resolve the frequency change induced by energy storage. Stein measured frequency changes induced by energy storage in a cryogenic niobium vacuum cavity and was not able to distinguish between surface effects and mechanical deformation [7]. Their results indicate coefficients about 2 orders of magnitude larger than those reported here. Our measurements were obtained using whispering gallery modes in a high Q factor, high purity monocrystalline sapphire dielectric resonator operated above 9 GHz, where the cavity wall has negligible influence on the mode frequency. Unloaded Q's as high as 4×10^9 at 4.2 K and 8×10^9 at 2 K [8] are possible in "whispering gallery" modes with the highest field confinement to the dielectric. These modes will be denoted by H_{c11} or E_{c11} according to whether they have predominantly magnetic or electric energy parallel to the resonator cylinder axis (i.e., quasi-TE or quasi-TM, respectively). The subscript c refers to the number of spatial field cycles in the circumferential direction, the radial and axial directions having only one variation.

For mode numbers $c \sim 11-15$ the electromagnetic fields are confined to a toroidal region centered on 0.9 of the radius, as depicted in Fig. 1, which has an effective volume about one-quarter of the total sapphire volume. Increasing the stored energy density causes the toroidal region to expand which results in a simultaneous increase (Δr) in the outside radius (r) and an increase $(\Delta\varepsilon)$ in the dielectric constant. The resulting fractional frequency

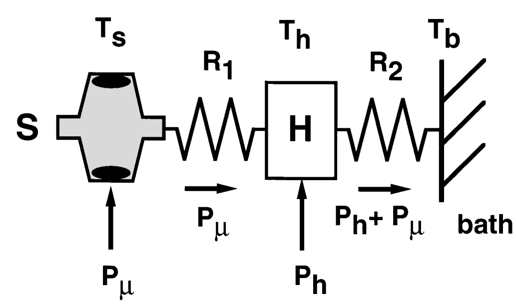

FIG. 1. Thermal circuit of the sapphire resonator (S) system comprising support structures (resistances, R_1 and R_2) and temperature control block (H), showing applied microwave (P_μ) and dc heater (P_h) powers. The darker shaded region of the sapphire resonator is the effective volume in which the electromagnetic fields are concentrated.

shift is

$$(\Delta f/f) = -[\Delta r/r + (1/2)\Delta\varepsilon/\varepsilon]. \quad (1)$$

If U is the total stored energy and V is the effective volume then (U/V) is, within a factor of order unity (a), the average electromagnetic stress, and

$$(\Delta\varepsilon/\varepsilon) \approx -a(U/V)(\varepsilon^{-1}\partial\varepsilon/\partial p). \quad (2)$$

Here $(\varepsilon^{-1}\partial\varepsilon/\partial p)$ is the constant describing the first derivative of ε with pressure, given by Link *et al.* [9] as -1.1×10^{-11} m^{-2}, the same (to two significant figures) for both crystal directions, perpendicular and parallel to the c axis. The stored energy density causes a strain $\sim U/VE$, where E is Young's modulus and resulting dimensional change:

$$(\Delta r/r) \approx a(U/V)(1/E). \quad (3)$$

Combining Eqs. (1)–(3), we obtain the fractional frequency shift which can be written as

$$f^{-1}\partial f/\partial U \approx -a[1 + (1/2)K_\varepsilon](EV)^{-1}, \quad (4)$$

where $K_\varepsilon = (-E\varepsilon^{-1}\partial\varepsilon/\partial p)$. Equation (4) is independent of resonator Q. In terms of the dissipated resonator power P $(= 2\pi fU/Q)$ and unloaded electrical Q factor, the fractional frequency shift is

$$f^{-1}\partial f/\partial P \approx -a[1 + (1/2)K_\varepsilon]Q(2\pi fEV)^{-1}, \quad (5)$$

which is proportional to resonator Q. Equation (5) is Braginsky's original expression [1], for low order modes in solid sapphire dielectric resonators coated with thin superconducting walls, with the numerical factor $a = \sqrt{2}/4$ for a TM_{010} mode and V denoting the total resonator volume. Braginsky claims that the constant K_ε characterizing the fractional change in dielectric constant per unit deformation lies in the range 1–10, which is consistent with the value (4.4) found from Link *et al.*'s data. The largest uncertainties in the above equations come from the effective volume V and averaging factor a. Calculations by Luiten [10] for whispering gallery modes give $V \sim 2 \times 10^{-6}$ m^3. Taking $a \sim 0.3$, $E = 4 \times 10^{11}$ Nm^{-2}, $K = 4.4$, and $f = 1 \times 10^{10}$ Hz in Eqs. (4) and (5), we estimate a fractional frequency shift of -1×10^{-12} μJ^{-1} or, equivalently, -3×10^{-11} mW^{-1} for a Q of 2×10^9. Thus to observe the radiation pressure effect we must have fractional frequency stability in our measurement system better than about 10^{-11}.

The fundamental limitations to the frequency stability of a macroscopic oscillator have been derived by Braginsky and Vyatchainin [11]. An oscillator can be considered as a resonator driven by an amplifier of noise temperature T_N. Thermal noise causes fractional frequency fluctuations $\Delta f/f_T$ according to the Townes-Schawlow formula: $\Delta f/f_T = (kT_N/2PQ^2\tau)^{1/2}$, where τ is the measurement integration time.

A second frequency fluctuation term arises through radiation pressure. As P increases, the amplitude of the electromagnetic radiation pressure and its fluctuations increase. This leads to a fluctuating mechanical deformation of the resonator, and degradation of frequency stability. Fluctuations in the occupation number N, δN, are given by $\delta N = (N\tau_e/\tau)^{1/2}$ in the high temperature limit, $kT > hf$, valid for microwave frequencies, where τ_e is the resonator relaxation time $Q/\pi f$. The corresponding power fluctuation δP is $\delta Nhf/\tau_e$ which, from Eq. (5), yields the fractional frequency fluctuation due to radiation pressure $\Delta f/f_P = -a(1 + 2^{-1}K_\varepsilon)Q(Phf/\tau)^{1/2}(2\pi fEV)^{-1}$.

Since $\Delta f/f_P$ has a power dependence inverse to that of $\Delta f/f_T$, there is an optimum power level which minimizes the sum of these two oscillator frequency fluctuations. The oscillator has a minimum frequency fluctuation $\Delta f/f_{\min}$ at an optimum power level, $P_{\rm opt} = A_n^{1/2}2^{-1/2}EVf(Q^2b)^{-1}$, where $A_n = kT_N/hf$ is the amplifier noise number and $b = a(1 + 2^{-1}K_\varepsilon)(2\pi)^{-1}$. The minimum frequency fluctuation is $\Delta f/f_{\min} = 2^{3/4}(bA_n^{1/2})^{1/2}(EV\tau/h)^{-1/2}$.

If the amplifier has an ideal quantum limited performance, $A_n = 1$, we find the oscillator quantum limit, $\Delta f/f_{\min} \approx 10^{-20}\tau^{-1/2}$, which may be independently deduced from an application of the Heisenberg energy uncertainty principle to the measurement of the minimum length change measurable in the lowest mechanical eigenmode of the resonator. For sapphire resonators of the type described here, $P_{\rm opt} \approx 10^{-2}$ W. In practice, the resonator driven by an amplifier with $A_n \gg 1$ elevates $P_{\rm opt}$ into an experimentally inaccessible regime. In addition, technical power instabilities cause radiation pressure fluctuations much larger than assumed above so that $\Delta f/f_P$ is then given by $(f^{-1}\partial f/\partial P)(\Delta P/P)$, where $\Delta P/P$ is the fractional power level stability of the oscillator [2].

We now describe observations which confirm Eqs. (4) and (5). We have been able to makes these measurements because of the extremely high electrical Q of sapphire, the good field confinement of whispering gallery modes, and the extremely good frequency stability of the resonator.

Experimental method.—The sapphire resonator was fabricated from Crystal Systems, Inc. "HEMEX" white high purity grade sapphire [12], which we will designate as CS3. In common with other HEMEX resonators previously studied, denoted CS1 and CS2 [3], a maximum of mode frequency as a function of temperature exists for all sapphire modes between 5 K and about 14 K. Operation near the temperature T_m of the frequency maximum considerably relaxes the temperature control requirements. We chose to study the H_{c11} modes because these exhibit a much flatter mode frequency maximum as a function of temperature than the E_{c11} modes [3] and have higher electrical Q factors at the frequency maximum because T_m is lower [4].

The geometry of the sapphire resonator has been described in detail elsewhere [8]. The sapphire element (S) is in the form of an optically polished spindle, shown schematically in cross section in the thermal circuit of Fig. 1. The sapphire is mounted inside a cylindrical gold-plated copper shield (inside diameter 8 cm and height 5 cm). Coupling to the microwave field is via superconducting solder-coated electric field probes. The main probe preferentially couples to the H_{c11} modes. Its coupling coefficient will be denoted by β. A second, weakly coupled probe is used for the loop oscillator configuration (described later). The darker shaded region in Fig. 1 is the effective volume in which the electromagnetic fields are concentrated. The sapphire resonator is housed in a permanently sealed vacuum enclosure attached to a liquid helium cryostat insert. It is isolated from the liquid helium bath via a high thermal impedance mechanical support (of thermal impedance $R_2 \sim 50$ K/W) and a second outer enclosure which can be evacuated. Temperature control to a precision of about 10^{-5} K is provided by a Lake Shore DRC-91CA temperature bridge connected to a four-terminal carbon glass thermometer and heater on a copper control block (H) which makes thermal contact between the support and the resonator shield. A thin layer of indium on one spindle end lowers the thermal resistance between the sapphire and shield, R_1 to about 5 K/W.

The microwave power dissipated in the resonator (P_μ) is measured by a substitution method. The control block temperature (T_h) and, because R_1 is much less than R_2, the sapphire resonator temperature (T_s) are servoed to be near to T_m. T_h is determined by the total power discharging through the thermal resistance which supports the resonator. This thermal current is made up of the power dissipated by the heater wire (P_h) and the power absorbed by the resonator (P_μ) from the incident microwave signal. By monitoring the change in the heater power required to maintain the resonator at a constant temperature in response to variations of the incident microwave power, we can determine the change in the dissipated microwave power. The long thermal time constant of the resonator and its vacuum pot necessitate a long sweep time to accurately calibrate the microwave power dissipated in the sapphire resonator as the incident microwave power is varied.

For the work described here we used the H_{c11} (quasi-TE$_{c1\delta}$) modes at 9.88, 10.57, 11.93, and 12.61 GHz for which the frequency-temperature dependence shows a maximum at 4.6 K and a corresponding unloaded Q factor of about 2×10^9. We conducted measurements at 5.3 K where the resonator frequency-temperature coefficient is about 4 Hz/K. Because the bath temperature ($T_b = 4.3$ K) is only 1 K below T_s, the maximum P_μ we can apply before temperature regulation ceases ($P_h = 0$) is only about 20 mW. The power flowing through R_1 elevates the average sapphire temperature up to 0.1 K, which has negligible effect on the mode frequency.

Two methods of mode center frequency readout are employed. In the first, the resonator mode center frequency is measured manually by probing the resonance with a frequency-swept microwave signal derived by mixing a high stability (about 10^{-12}) fixed frequency source with a high resolution synthesizer. The spurious frequency signals created in the mixing process are filtered from the probing microwave signal using a tunable bandpass filter to prevent interference with the desired signal. In this method the frequency resolution is about 5×10^{-12}.

In the second method, used only in the case of the H_{1411} mode, the frequency is measured to much higher precision, by using the resonator as a feedback element of a microwave loop oscillator. This loop oscillator is further stabilized by using an active Pound frequency discriminator system giving a fractional frequency stability of $\sim 10^{-15}$ [4]. This microwave frequency is double heterodyned down to audio frequencies using a second high stability loop oscillator and an rf synthesizer so that it can be counted with a conventional frequency counter.

All of the microwave measurements system components are at room temperature, except for the ferrite circulator (one per coupling probe) and isolators (two per transmission line to and from the circulator) that are placed near the resonator to minimize frequency pulling effects.

Experimental observations and discussion.—The manually measured frequency power dependence $f(P)$ of the resonator at 5.3 K is shown in Fig. 2 for the modes at 9.9, 10.6, 11.9, and 12.6 GHz. It can be seen that in all cases the slope ($\partial f/\partial P$) is negative, and the fit is linear with about a 5% statistical error. The lack of curvature at high power indicates that the effect of the thermal resistance R_1 is negligible. The magnitude of $-\partial f/\partial P$ mode parameters and the inferred fractional frequency shift with stored energy, $-1/f \partial f/\partial U$, are summarized in Table I. The latter quantity is obtained from $\partial f/\partial P$ by multiplying by $2\pi/Q$. The coupling in all the modes is sufficiently high that a large

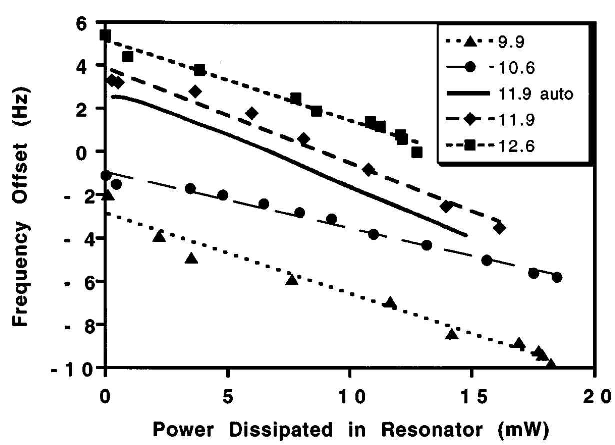

FIG. 2. The measured mode frequency as a function of dissipated power for the modes at 9.9, 10.6, 11.9, and 12.6 GHz. The dashed lines represent lines of best fit through the experimental points of the manual measurements. "11.9 auto" is a fully automated measurement of the 11.9 GHz (H_{1411}) mode using a loop oscillator.

TABLE I. Mode parameters, measured slope $-\partial f/\partial P$, and the inferred fractional frequency shift with stored energy, $-1/f\,\partial f/\partial U$, are summarized for the H_{c11} modes studied from 9.9 to 12.6 GHz.

Mode freq. (GHz)	9.88	10.57	11.25	11.93	12.61
Mode No., c	11	12	13	14	15
$-\partial f/\partial P$ (Hz/mW) $\pm$ 5%	0.37	0.26	$\cdots$	0.45	0.41
Unloaded $Q(10^9)$ $\pm$ 5%	2.3	1.75	0.6	2.4	2.2
Coupling	4.7	0.28	0.23	0.31	0.11
$-1/f\,\partial f/\partial U$ $(10^{-12}/\mu\text{J})$ $\pm$ 8%	1.0	0.92	$\cdots$	1.2	1.2

fraction of incident power, $4\beta/(1+\beta)^2$, is absorbed by the sapphire. We did not attempt to include the mode at 11.25 GHz because its resonance curve is distorted and its unloaded Q considerably diminished by a nearby low Q mode. The measured slope of fractional frequency shift with stored energy $1/f\,\partial f/\partial U$ is essentially constant from 9.9 GHz to 12.6 GHz, varying only from -1.0 ± 0.1 to $-1.2 \pm 0.1 \times 10^{-12}/\mu\text{J}$. This implies from Eq. (1) that V is nearly constant from 9.9 GHz to 12.6 GHz. Intuitively, one expects a slight increase with mode frequency as the fields become concentrated in a smaller volume. The independence of frequency-power dependence $f(P)$ on the coupling indicates that the observed effect depends only on the unloaded resonator, hence internal dissipation, and is not some frequency pulling effect.

For the 11.93 GHz mode (-0.45 Hz/mW) there is fair agreement with previous measurements [5]: -0.6 and -0.4 Hz/mW for HEMEX resonators (CS1 and CS2) of identical dimensions and an unloaded Q of about 2×10^9 [5].

An automated measurement (Fig. 2) using a loop oscillator system gives an accurate measurement of the slope for the 11.93 GHz mode. This measurement gives the same slope as the previous manual measurement. In the low power region (below 0.6 mW) the deviation from linearity is caused by residual offset voltages in the frequency control servo which become more important at low gain.

The measured values $-1/f\,\partial f/\partial U$ and $-\partial f/\partial P$ are in agreement with the rough estimates based on Eqs. (4) and (5). This, together with the sign and linearity of $f(P)$ and the correct Q dependence, are taken as strong evidence that the phenomenon observed here is indeed the radiation pressure term of Braginsky [1].

In conclusion, we report observations of radiation pressure induced lattice expansion in a solid dielectric resonator. The measurements were made on high Q-factor quasi-TE whispering gallery modes from 9.9 to 12.6 GHz in a monocrystalline sapphire resonator at liquid helium temperatures. The fractional frequency shift is in the range -1.0 to -1.2×10^{-12} per μJ of energy stored in the resonator. This result is consistent in sign, magnitude, and linearity with the radiation pressure induced lattice expansion term predicted by Braginsky.

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