$\Delta I = 4$ Bifurcation in Ground Bands of Even-Even Nuclei and the Interacting Boson Model

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We find a $\Delta I = 4$ bifurcation phenomenon in many ground state rotational bands by performing a staggering parameter analysis, which has been employed often for superdeformed bands. We find also that a small modification of the intrinsic state of the interacting boson model wave function in the *sd*-boson space, which has been used extensively for ground bands, is able to produce the regular $\Delta I = 4$ bifurcation pattern. Adding a g boson, we are able to generate even an irregular pattern in the $\Delta I = 4$ bifurcation phenomenon, which seems to be observed in some ground bands. [S0031-9007(97)04011-8]

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Recently, extensive studies of superdeformed bands revealed a $\Delta I = 4$ bifurcation phenomenon by calculating the staggering parameter [1,2],

$$\Delta E_\gamma(I) = \frac{3}{8}\{E_\gamma(I) - \frac{1}{6}[4E_\gamma(I-2) + 4E_\gamma(I+2) - E_\gamma(I-4) - E_\gamma(I+4)]\} \tag{1}$$

introduced by Cederwall *et al.* [2]. Here, $E_\gamma(I)$ is the transition energy from a spin state with I to $I-2$. If the rotational energy follows the $I(I+1)$ law, then $\Delta E_\gamma(I) = 0$. However, in some superdeformed bands, rotational energies are irregular and $\Delta E_\gamma(I)$ shows a zigzagging pattern between neighboring spin states.

Motivated by this finding in the superdeformed bands, we focus our attention on the ground state bands of the even-even rotational nuclei. Since there are many precise experimental data on the ground state bands, we have made a staggering parameter analysis of known ground state bands up to very high spins with the bandcrossing spin more than 16, $I_c > 16$, over the nuclear chart. To our surprise, we found the $\Delta I = 4$ bifurcation phenomenon in many ground bands (with $I_c > 16$) for the following nuclei: ^{162}Dy, ^{174}Yb, ^{230,232}Th, $^{232\sim238}$U, ^{236,238,242,244}Pu, and ^{248}Cm. Those of the Hf and Er isotopes are quite smooth (no bifurcation). We show in Fig. 1 two examples of the dynamical moment of inertia of ^{230}Th and ^{232}Th and the corresponding staggering parameters. We see clearly the bifurcation patterns in these nuclei. The size of this anomalous behavior is about 0.3 keV, which is similar to those in the superdeformed bands [1,2]. Interesting to note is that the signs of the zigzagging behavior in these two neighboring even-even isotopes are opposite.

The observation of $\Delta I = 4$ bifurcation in superdeformed bands has motivated various theoretical studies. Filbotte *et al.* suggested an explanation based on a fourfold symmetry (C_4 symmetry) in the nuclear shape [1]. Theoretical works were then made by including some C_4 symmetry term in the Hamiltonian, with a rotation axis either along or vertical to the symmetry axis [4–6]. The interacting boson model (IBM) with g bosons was also employed to reproduce the Hamiltonian with the C_4 symmetry in the geometrical model by taking the limit of the boson number $N \to \infty$ [7]. There were also theoretical studies, where $\Delta I = 4$ bifurcation can arise from the mixing of two bands near the yrast line [8] without the explicit use of the C_4 symmetry or from an intrinsic vortical motion [9].

In order to study this behavior theoretically, we take the simplest viewpoint and employ the interacting boson model, which is extensively used for the analysis of ground state bands [10]. In the SU(3) limit of s and d bosons, the intrinsic state reads [11]

$$\Phi_{\rm int} = \frac{1}{\sqrt{N!(1+\beta^2)^N}}[s^\dagger + \beta d_0^\dagger]^N|0\rangle, \tag{2}$$

where $\beta = \sqrt{2}$. The standard angular momentum projection out of this intrinsic state provides only a smooth rotational band. $s^\dagger$ and $d_0^\dagger$ are the creation operators of the s and d bosons ($d_0^\dagger$ denotes the creation operator of d boson

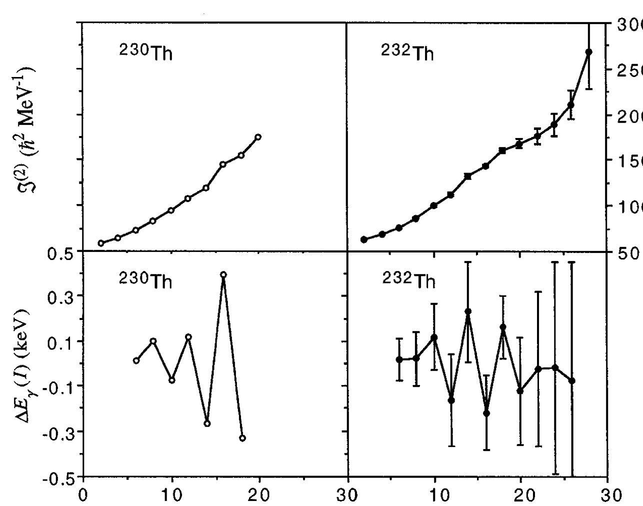

FIG. 1. Dynamical moments of inertia (upper) of the ground state band in ^{230}Th and ^{232}Th [3] and corresponding staggering parameters (lower) plotted as a function of nuclear spin. Experimental errors are not provided for ^{230}Th except for $E_\gamma(2\to0)$, $E_\gamma(4\to2)$, and $E_\gamma(6\to4)$ with errors of 0.02, 0.02, and 0.5 keV, respectively, and, hence, we are not able to provide error bars in the staggering parameters.

with the magnetic quantum number $m = 0$) and N is interpreted as the number of bosons. This intrinsic state (2) is constructed by the standard group theory of SU(3) (see, e.g., Ref. [11]).

In Ref. [12], Ginocchio and Kirson get this intrinsic state by using the variation before projection (VBP) method, which is extensively used in the study of nuclear structure [13]. They prove that any states in IBM can be projected out from general intrinsic states,

$$|\Phi(N,\beta,\gamma)\rangle = \frac{1}{\sqrt{N!(1+\beta^2)^N}} \times \{s^\dagger + \beta[\cos\gamma d_0^\dagger + \sin\gamma(d_2^\dagger + d_{-2}^\dagger)/\sqrt{2}]\}^N|0\rangle .$$

This intrinsic state (2) is obtained by minimizing $\langle\Phi(N,\beta,\gamma)|H|\Phi(N,\beta,\gamma)\rangle$ energy surface of the SU(3) Hamiltonian, i.e., the equilibrium values $\gamma = 0$ and $\beta \approx \sqrt{2}$ in $|\Phi(N,\beta,\gamma)\rangle$. All of the rotational states (with angular momenta I; $I = 0,2,\ldots,2N$) of the $(2N,0)$ (ground state band) representation of SU(3) can be projected out from (2). It means that the VBP gives the exact rotational energy in the case of the SU(3) limit. In general, the VBP should also give a very good approximation for the ground state bands, at least for the low-lying states [13]. Following their technique and considering the interaction $\nu_2[(s^\dagger d^\dagger)^2(\tilde{d}\tilde{d})^2 + \text{H.c.}]$ $(\nu_2 > 0)$ in a general IBM Hamiltonian, we can give the equilibrium at $\gamma = 0$ and at some β by minimizing the same energy surface $\langle\Phi(N,\beta,\gamma)|H|\Phi(N,\beta,\gamma)\rangle$, but β is a solution of the equation $d\langle\Phi_{\text{int}}|H|\Phi_{\text{int}}\rangle/d\beta = 0$. It means that $|\Phi(N,\beta,\gamma)\rangle$ is reduced into the intrinsic state (2) for ground state bands, where the value of β may be arbitrary, if the parameters in the IBM Hamiltonian take general values deviating from the SU(3) limit. As in the case of the SU(3) limit, we then project our rotational states with good angular momenta from the intrinsic state (2) [12,13].

The expectation value of the IBM Hamiltonian for the members of the ground state band is written as

$$E_I(\beta) = \frac{\int_0^\pi d\cos\theta P_I(\cos\theta)\langle\Phi_{\text{int}}|H\exp(i\theta\hat{J}_y)|\Phi_{\text{int}}\rangle}{\int_0^\pi d\cos\theta P_I(\cos\theta)\langle\Phi_{\text{int}}|\exp(i\theta\hat{J}_y)|\Phi_{\text{int}}\rangle} . \tag{3}$$

Here $P_I(\cos\theta)$ is the Legendre polynomial and $\hat{J}_y$ is the y component of the angular momentum operator.

We first show that the IBM Hamiltonian can provide β at any value within the VBP. For instance, when the IBM Hamiltonian has the form

$$H = \varepsilon_d d^\dagger\tilde{d} + \nu_2[(d^\dagger d^\dagger)^2\tilde{d}s + \text{H.c.}] + C_0(d^\dagger d^\dagger)^2(\tilde{d}\tilde{d})^0, \tag{4}$$

we find various values of β by changing C_0. $\beta = 2.67$, if $E_d = 5$ keV, $\nu_2 = 40$ keV, and $C_0 = -84$ keV. We show in Fig. 2 the outcome of the numerical results on $\Delta E_\gamma(I)$ by changing the value of C_0, which result in various values for β. It is interesting to note that the bifurcation behavior appears and increases as β deviates from $\sqrt{2}$. Also of interest is the fact that, by changing the boson number from $N = 12$ to 11, the zigzag pattern of $\Delta E_\gamma(I)$ is reversed, as shown in Fig. 3.

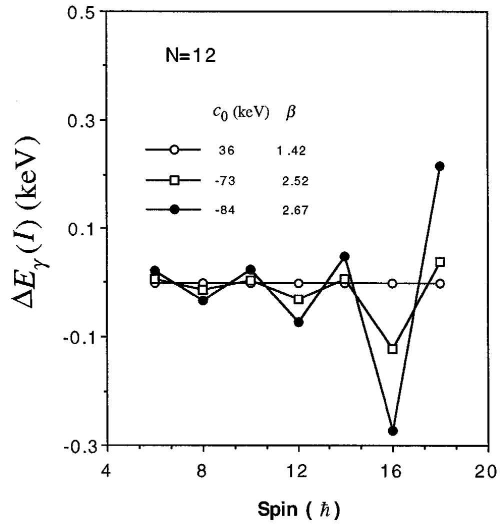

FIG. 2. The staggering parameter, $\Delta E_\gamma(I)$ as a function of the nuclear spin for various β^2, which is the ratio of the s and d bosons. The zigzagging behavior increases with the values of β, which is determined by various values of C_0. The parameters used are $\varepsilon_d = 5$ keV and $\nu_2 = 40$ keV in (4). The boson number is $N = 12$.

To understand this behavior, we show in the inset of Fig. 3 the overlap kernel,

$$\mathcal{N}(\theta) = \langle\Phi_{\text{int}}|\exp(i\theta\hat{J}_y)|\Phi_{\text{int}}\rangle = \{(1 - \beta^2/2 + 3\beta^2\cos^2\theta/2)/(1+\beta^2)\}^N . \tag{5}$$

If $\beta = \sqrt{2}$, $\mathcal{N}(\theta) = \cos^{2N}\theta$, and there appear two peaks at $\theta = 0$ and π. If $\beta > \sqrt{2}$, there appears a new very small

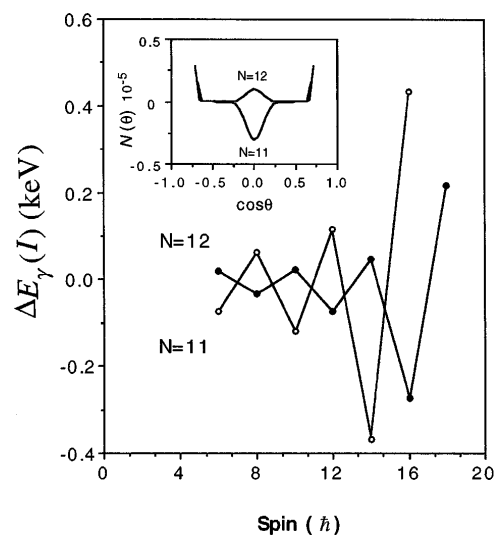

FIG. 3. The calculated staggering parameters $\Delta E_\gamma(I)$ as a function of the nuclear spin for $N = 11$, 12. The parameters used are the same as in Fig. 2 and $C_0 = -84$ keV. The inset shows the overlap kernels. The kernel goes up to 1 smoothly at $\theta = 0$ and π. The values of β determined by the VBP method are 2.669 for $N = 11$ and 2.67 for $N = 12$. The signs of the zigzagging behavior are opposite by changing the boson number by 1.

peak at $\theta = \pi/2$ besides the two original peaks with the overlap kernel equal to 1 at $\theta = 0$ and π. We draw the results of the overlap kernels for $N = 11$ and 12 in the inset of Fig. 3, which shows that the signs of the small peaks of the two cases for $N = 11$ and 12 are opposite. The value of the overlap kernel is about 10^{-6} at $\theta = \pi/2$. This tiny peak appears at $\theta = \pi/2$ and changes sign with $(-)^N$ for $\beta > \sqrt{2}$, because $1 - \beta^2/2$ in the numerator of $\mathcal{N}(\theta)$ in (5) becomes negative while the $\cos^2\theta$ term is throughout positive and becomes zero at $\theta = \pi/2$. Furthermore, this tiny peak with maxima $\mathcal{N}(\pi/2) = (-)^N[(B^2 - 2)/(2 + 2\beta^2)]^N$ fades away as $N \to \infty$. This appearance of the tiny peak in $\mathcal{N}(\theta)$ is related to the staggering phenomenon in the rotational band.

Knowing the property of the norm kernel, we can derive the approximate expression of the energy $E_I(\beta)$ in (3) following the method used in deriving the $I(I + 1)$ low [14]. The energy $E_I(\beta)$ can be expressed as

$$E_I(\beta) - E_0(\beta) = -G(-)^N[P_I(0) - P_0(0)] + E_{\rm rot}(I). \tag{6}$$

The smooth rotational energy $E_{\rm rot}(I)$ arises from the two peaks at $\theta = 0$ and π and has the $I(I + 1)$ law behavior in the first order approximation [14]. The factor G for the Hamiltonian (4) is given as

$$G = 2NG_d|\mathcal{N}(\pi/2)|/\Gamma, \tag{7}$$

where $G_d = [\varepsilon_d(\beta^2 - \beta^4/2) - (N - 1)(\sqrt{8/7}\nu_2\beta^3 - 2C_0\beta^4/5)]/(1 + \beta^2)^2$ and Γ being the width of the peaks at $\theta = 0$ and π: $\Gamma = 2(1 + \beta^2)/(3\beta^2 N)$. The new tiny peak at $\theta = \pi/2$ makes the zero-point value $P_I(0)$ of the Legendre polynomial come out, where $P_I(0) = (-)^{I/2}(I - 1)!!/I!!$ is responsible for the staggering effect [8]. Meanwhile, the tiny peak also makes the factor $(-)^N$ appear, which is the origin of the different signs of two neighboring nuclei. The boson number in the factor $(-)^N$ plays a role similar to α_4 in Ref. [1]. If we use the value $\varepsilon_d = 5$ keV, $\nu_2 = 40$ keV, and $C_0 = -84$ keV, which provides the smooth rotational energy, we find the amplitude of the staggering to be about 0.1 keV. In general, the effect from a specific Hamiltonian appears only in the factor G_d. It is easily proven that the form of (7) is valid for all kinds of Hamiltonians which can drive $\beta > \sqrt{2}$. Only the effect of different Hamiltonians is to provide the different forms for G_d. The staggering term in (6) fades away as $N \to \infty$ due to $G \to 0$. Numerical calculation also proves this point. From the above analysis for the overlap kernel, it is not difficult to understand the bifurcation phenomena. The tiny peak $\theta = \pi/2$ is the essential origin of the $\Delta I = 4$ bifurcation. In order to create the tiny peak, only two conditions are required: $\beta > \sqrt{2}$ and the boson number being finite. For a general Hamiltonian (4.4) in [12] with nine free parameters, there are many sets of parameters, which can give the same value of β under the VBP principle. This seems to imply that the staggering effect does not arise from the specific dynamical property. Since the tiny peak or the factor G disappears completely as $N \to \infty$, no matter how big β is, the most important source of the bifurcation phenomenon is the fact that the boson number is finite.

The use of the general IBM Hamiltonian provides $\beta > \sqrt{2}$ as required for the $\Delta I = 4$ bifurcation. Ginnocchi and Kirson gave a relation between β of the IBM and the β deformation of the geometrical model studied extensively by Bohr and Mottelson (BM) under a certain approximation as $\beta(\mathrm{BM}) \leq 1.18n\beta/A$ [12], where $n = 2N$ is the valence nucleon number. If we take $\beta(\mathrm{BM}) \sim 0.3$, $N \sim 11$, and $A \sim 230$, then we find $\beta \geq 2.66$. Warner and Casten [15] also gave a slightly different relation, which is $\beta = A\beta(\mathrm{BM})/n$. For those nuclei mentioned above, $A/n = 8$–10 and $\beta(\mathrm{BM}) \sim 0.3$, then $\beta = 2.4$–3. These values are very close to the value determined by minimizing the Hamiltonian (4). For typical values of β, one has, in rare-earth nuclei (i.e., for nucleus near ^{156}Gd) $\beta \approx 6.5\beta(\mathrm{BM})$ that coincides with $\beta \approx 6.33\beta(\mathrm{BM})$ given by Iachello and Arima [16]; in the actinide region, $\beta \approx 9.2\beta(\mathrm{BM})$. In all cases, $\beta > \sqrt{2}$ is $\beta(\mathrm{BM}) > 0.25$.

We discuss here the validity of the VBP method. For this purpose, we take the full numerical diagonalization method by using the computer code PHINT made by Scholten [17]. We found almost perfect reproduction on the rotational energies of the ground state band. Concerning the staggering effect (very small quantity), the full diagonalization provides an even larger $\Delta I = 2$ staggering. Since we know from the SU(3) limit, the VBP provides a slightly smaller β value $\beta = \sqrt{2}[1 - 3/8N + O(1/N^2)]$, as compared with the exact one, $\beta_{\rm exact} = \sqrt{2}$, we modify slightly the β value by a few percent, which does not disturb the rotational spectra at all. We again get good agreement on the staggering phenomenon. We shall discuss this conclusion in more detail in a future publication [18].

In the experimental data, there exist cases where the zigzagging pattern becomes irregular at some spin, such as ^{174}Yb at spin 14, ^{232}Th at 24, ^{234}U at 16, ^{236}U at 18, and ^{242}Pu at 16 which is shown in Fig. 4 as an example. As demonstrated above, this irregular pattern is not expected in the intrinsic states made of the *sd* bosons. Hence, we add the g_0 boson to the *sd* space as

$$\Psi_{\rm int} = \frac{1}{\sqrt{N!(1 + \beta^2 + \varepsilon^2)}}[s^\dagger + \beta d_0^\dagger + \varepsilon g_0^\dagger]^N|0\rangle. \tag{8}$$

and do the same angular momentum projection as in the case of the *sd* space. We plot the staggering parameter of the Hamiltonian $H_{gd} = C_{gd}[(g^\dagger g^\dagger)^4(\tilde{d}\tilde{d})^4 + \mathrm{H.c.}]$ in the bottom of Fig. 4 to compare with the corresponding experimental result of ^{242}Pu. In the calculation, we took $C_{gd} = 4040.0$ keV. To simplify and view the contribution of the g_0 boson, we set $\beta = \sqrt{2}$. We clearly see the irregular pattern which may correspond to what the experimental data suggest, although the experimental errors ought to be reduced. The purpose of showing this comparison is merely to demonstrate that the IBM with inclusion of the g boson is able to generate this

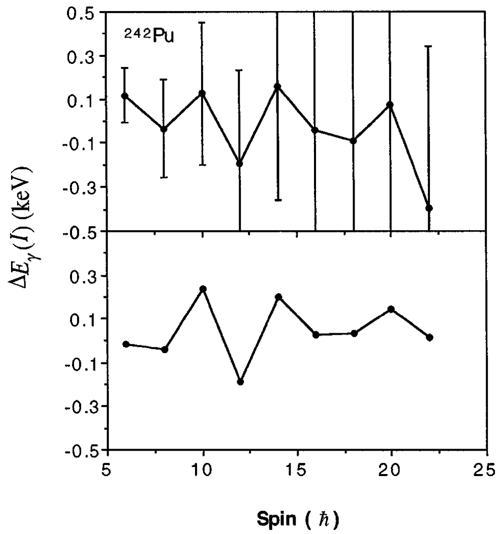

FIG. 4. The staggering parameter for ^{242}Pu [3] as a function of the nuclear spin. The calculated staggering parameter is obtained by adding the g boson with $N = 19$ and $\varepsilon^2 = 1.8$, and $C_{gd} = 4020.0$ keV, while keeping $\beta^2 = 2$.

irregular behavior. If we plot the overlap kernel, we see two more peaks at $\theta_m(\varepsilon)$ and $\pi - \theta_m(\varepsilon)$, which depend on the value of ε, in addition to those at $\theta = 0$, $\pi/2$, and π. So, roughly, the zigzagging part of the excitation energy is proportional to $P_I(\cos\pi/2) + 2P_I(\cos\theta_m)$, which represents a sum of a regular staggering pattern provided by $P_I(\cos\pi/2)$ and an irregular part provided by $2P_I(\cos\theta_m)$. Hence, we are able to describe even more irregular patterns in the IBM with the g boson.

We remark on the effect of the bandcrossing on the staggering parameter. The bandcrossing can also give finite contributions to the staggering parameter, which is usually very large (of order of 10 keV) and concentrated around the bandcrossing spin due to the definition of the staggering parameter. This, however, has nothing to do with the $\Delta I = 4$ bifurcation phenomenon considered in this study. In order to avoid this trivial anomaly, we have taken the rotational bands with the bandcrossing spin more than 16 ($I_c \geq 16$) and have stopped the analysis before this anomaly sets in.

In conclusion, we have analyzed systematically the ground state bands of all the nuclei using the staggering parameter. We have found the $\Delta I = 4$ bifurcation phenomenon in many cases as found in the other rotational bands. We have even found cases where the zigzagging pattern becomes irregular. We have then taken the interacting boson model and modified slightly the intrinsic wave function of the SU(3) limit of IBM by using the VBP principle. We have also found, in the experimental analysis, a case where the zigzagging pattern becomes opposite of a certain spin and they are given a natural explanation within IBM.

The $\Delta I = 4$ bifurcation phenomenon seems a common feature of rotational bands and seems not to depend strongly on the dynamical properties of a system. In our theory, the staggering effect depends only on the value of β and the boson number being finite. It means that the effect is related only to the geometrical shape of a nucleus since β is a renormalized geometrical quadrupole deformation.

In this connection it is very interesting to note that the $\Delta I = 4$ bifurcation phenomenon is also identified recently in the rotational spectra of diatomic molecules [19]. The finding of the present Letter may also be applicable to the staggering pattern in the negative parity bands of the even-even nuclei and in the rotational bands in odd mass nuclei [20]. We are studying these totally different systems in terms of the IBM by changing the contents of the intrinsic state.

We hope our study induces experiments to study more carefully the ground state bands and to provide more examples of $\Delta I = 4$ bifurcation. In addition, it is interesting to apply the modified IBM to the superdeformed bands.

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