Induced Parity-Breaking Term at Finite Temperature

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We compute the exact induced parity-breaking part of the effective action for $(2 + 1)$ massive fermions in QED_3 at finite temperature by calculating the fermion determinant in a particular background. The result confirms that gauge invariance of the effective action is respected even when large gauge transformations are considered. [S0031-9007(97)04004-0]

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Because of its relevance in both field theory and condensed matter physics, much effort has been devoted in the past decade to the study of three-dimensional gauge theories coupled to matter. An important ingredient in these theories is the parity anomaly [1] which induces, through fluctuation of massive Fermi fields, a Chern-Simons term in the effective action of the gauge field [2].

As originally stressed in [1], a fundamental property of the Chern-Simons (CS) action is the existence of a quantization law: Because of the noninvariance of the CS term under gauge transformations of "nonzero" winding number, the coefficient which appears in front of the (non-Abelian) Chern-Simons three form S_{CS} should be quantized so that $\exp(iS_{\mathrm{CS}})$ is single valued. Even in the Abelian case, "large" gauge transformations come into play whenever the theory is formulated in an appropriately compactified manifold [3,4] and in that case the quantization law also holds. Putting together all these facts, one can state that, in three-dimensional gauge theories with Fermi fields, calculations of the effective action for the gauge field using gauge-invariant regularizations lead to a parity anomaly which manifests through the occurrence of a CS term with a quantized coefficient which depends on the number of fermion species.

A natural question raised when the analysis of three-dimensional gauge theories was extended to the case of finite temperature was whether quantization of the CS coefficient induced by fermion fluctuations survives the effects of temperature or is smoothly renormalized. The question was originally discussed in [5], where it was argued that the coefficient of the CS term in the effective action for the gauge field should remain unchanged at finite temperature. Contrasting with this analysis, perturbative calculations for both relativistic and nonrelativistic theories, Abelian and non-Abelian, have yielded induced actions with CS coefficients which are smooth functions of the temperature [6–15], this seeming to signal a kind of gauge anomaly at finite temperature.

The problem of renormalization of the CS coefficient induced by fermions at $T \neq 0$ was revisited in Refs. [16,17], where it was concluded that, on gauge invariance grounds and in perturbation theory, the effective action for the gauge field cannot contain a smoothly renormalized CS coefficient at nonzero temperature. More recently, the exact result for the effective action of a $(0 + 1)$ analog of the CS system [18] as well as a zeta-function analysis of the fermion determinant at $T \neq 0$ in the $(2 + 1)$ model [19] have explicitly shown that although the perturbative expansion leads to a smooth temperature-dependent and hence nonquantized CS coefficient, the complete effective action can be made gauge invariant; the induced CS term's noninvariance revealed by perturbation theory being compensated by nonlocal contributions to the effective action.

Originally [2], the parity anomaly for fermions at $T = 0$ was analyzed by considering a particular gauge field background configuration which allowed the closed computation of the anomalous part of the fermion current. In the same vein, we compute in this work the induced parity-breaking part of the effective action for three-dimensional massive fermions in QED_3 at finite temperature by considering a particular gauge field configuration which allows the attainment of a closed exact result for the fermion determinant. Our result confirms that gauge invariance, even under large gauge transformations, is respected, and at the same time reproduces in the appropriate limits the perturbative and zero-temperature results.

We define the total effective action $\Gamma(A)$, as usual, by the formula

$$e^{-\Gamma(A,M)} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \times \exp\left[-\int_0^\beta d\tau \int d^2x\, \bar{\psi}(\not\partial + ie\not{A} + M)\psi \right]. \tag{1}$$

We are using Euclidean Dirac's matrices in the representation $\gamma_\mu = \sigma_\mu$, and $\beta = \frac{1}{T}$ is the inverse temperature. The label 3 is used to denote the Euclidean time coordinate τ. The fermionic (gauge) fields in (1) obey antiperiodic (periodic) boundary conditions in the timelike direction. We are concerned with the mass-dependent parity-odd piece Γ_{odd} of Γ, which, as a parity transformation changes the sign of the mass term (the only odd term under parity in the Euclidean action), can be obtained as follows:

$$2\Gamma_{\mathrm{odd}}(A,M) = \Gamma(A,M) - \Gamma(A,-M). \tag{2}$$

In any gauge-invariant regularization scheme there is also a mass-independent (and temperature-independent) contribution (the parity anomaly) which corresponds to a CS term with a coefficient such that it changes in multiples of $i\pi$ under large gauge transformations [19,20].

The calculation of (2) for *any* gauge field configuration is not something we can do exactly. Instead of making a perturbative calculation dealing with a small but otherwise arbitrary gauge field configuration, we shall consider a restricted set of gauge field configurations which can, however, be treated exactly. Moreover, as we want to make a calculation which preserves the symmetry for gauge transformations with nontrivial winding around the time coordinate, any approximation assuming the smallness of A_3 could put this symmetry in jeopardy.

A convenient class of configurations from this point of view is that of time-independent magnetic fields in a gauge such that

$$A_3 = A_3(\tau), \qquad A_j = A_j(x)\,(j = 1,2), \tag{3}$$

namely, A_3 is only a function of τ, and A_j is independent of τ. Under these assumptions, we see that the only τ dependence of the Dirac operator comes from A_3. This dependence can, however, be erased by a redefinition of the integrated fermionic fields. The set of allowed gauge transformations in the imaginary time formalism is defined in the usual way:

$$\begin{aligned}\psi(\tau,x) &\to e^{-ie\Omega(\tau,x)}\psi(\tau,x),\\ \bar\psi(\tau,x) &\to e^{ie\Omega(\tau,x)}\bar\psi(\tau,x),\\ A_\mu(\tau,x) &\to A_\mu(\tau,x) + \partial_\mu\Omega(\tau,x),\end{aligned} \tag{4}$$

where $\Omega(\tau,x)$ is a differentiable function vanishing at spatial infinity ($|x| \to \infty$), and whose time boundary conditions are chosen in order not to affect the fields' boundary conditions. It turns out that $\Omega(\tau,x)$ can wind an arbitrary number of times around the cyclic time dimension,

$$\Omega(\beta,x) = \Omega(0,x) + \frac{2\pi}{e}n, \tag{5}$$

where n is an integer which labels the homotopy class of the gauge transformation.

As we are interested in evaluating the fermionic determinant in a gauge-invariant way,

$$\begin{aligned}\det(\partial\!\!\!/ + ieA\!\!\!/ + M) = \int \mathcal{D}\psi\,\mathcal{D}\bar\psi\\ \times \exp\biggl\{-\int_0^\beta d\tau \int d^2x\\ \times \bar\psi(\partial\!\!\!/ + ieA\!\!\!/ + M)\psi\biggr\},\end{aligned} \tag{6}$$

we can always perform a gauge transformation in order to pass to an equivalent expression, where the gauge field $A'_\mu = A_\mu + \partial_\mu\Omega$ is constant in time. For the particular set of configurations (3), such a transformation renders A'_3 constant. We see that there is a family of Ω's achieving this while respecting the boundary conditions (5),

$$\begin{aligned}\Omega(\tau) = -\int_0^\tau d\tilde\tau\,A_3(\tilde\tau)\\ + \left(\frac{1}{\beta}\int_0^\beta d\tilde\tau\,A_3(\tilde\tau) + \frac{2\pi n}{e\beta}\right)\tau,\end{aligned} \tag{7}$$

where n is an arbitrary integer. The freedom to choose n could be used to further restrict the values of the constant A'_3 to a finite interval. In this sense, the value of the constant in such an interval is the only "essential," i.e., gauge invariant, A_3-dependent information contained in the configurations (3), describing the holonomy $\int_0^\beta d\tilde\tau\,A_3(\tilde\tau)$. However, we will limit ourselves to small gauge transformations ($n = 0$) in order to avoid any assumption about large gauge invariance of the fermionic measure in (6) and safely discuss the effect of large gauge transformations on the final results. Thus the constant field A'_3 simply takes the mean value of $A_3(\tau)$, $\tilde A_3 = \frac{1}{\beta}\int_0^\beta d\tau\,A_3(\tau)$. Note that the spatial components of A_μ remain τ independent after this transformation.

It is convenient to perform a Fourier transformation on the time variable for ψ and $\bar\psi$, since the Dirac operator is now invariant under translations in that coordinate,

$$\begin{aligned}\psi(\tau,x) &= \frac{1}{\beta}\sum_{n=-\infty}^{n=+\infty} e^{i\omega_n\tau}\psi_n(x),\\ \bar\psi(\tau,x) &= \frac{1}{\beta}\sum_{n=-\infty}^{n=+\infty} e^{-i\omega_n\tau}\bar\psi_n(x),\end{aligned} \tag{8}$$

where $\omega_n = (2n+1)\frac{\pi}{\beta}$ is the usual Matsubara frequency for fermions. Then the Euclidean action is written as an infinite series of decoupled actions, one for each Matsubara mode,

$$\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int d^2x\,\bar\psi_n(x)[\not{d} + M + i\gamma_3(\omega_n + e\tilde A_3)]\psi_n(x), \tag{9}$$

where $\not{d} = \gamma_j(\partial_j + ieA_j)$ is the Dirac operator corresponding to the spatial coordinates and the spatial components of the gauge field.

As the action splits up into a series and the fermionic measure can be written as

$$\mathcal{D}\psi(\tau,x)\mathcal{D}\bar\psi(\tau,x) = \prod_{n=-\infty}^{n=+\infty}\mathcal{D}\psi_n(x)\mathcal{D}\bar\psi_n(x), \tag{10}$$

the $(2+1)$ determinant is an infinite product of the corresponding $(1+1)$ Euclidean Dirac operators,

$$\det(\partial\!\!\!/ + ieA\!\!\!/ + M) = \prod_{n=-\infty}^{n=+\infty}\det[\not{d} + \rho_n e^{i\gamma_3\phi_n}], \tag{11}$$

where we have also defined

$$\rho_n = \sqrt{M^2 + (\omega_n + e\tilde{A}_3)^2}, \qquad \phi_n = \arctan\left(\frac{\omega_n + e\tilde{A}_3}{M}\right). \tag{12}$$

Explicitly, the $(1 + 1)$ determinant for a given mode is a functional integral over $(1 + 1)$ fermions,

$$\det[\not{d} + \rho_n e^{i\gamma_3\phi_n}] = \int \mathcal{D}\chi_n \mathcal{D}\bar{\chi}_n \times \exp\left\{-\int d^2x\, \bar{\chi}_n(x) \times (\not{d} + \rho_n e^{i\gamma_3\phi_n})\chi_n(x)\right\}. \tag{13}$$

We now realize that the change of fermionic variables,

$$\chi_n(x) = e^{-i(\phi_n/2)\gamma_3}\chi_n'(x), \qquad \bar{\chi}_n(x) = \bar{\chi}_n'(x)e^{-i(\phi_n/2)\gamma_3}, \tag{14}$$

makes the action in (13) independent of ϕ_n. Concerning the fermionic measure, it picks up an anomalous Fujikawa Jacobian [21] so that one ends with

$$\det[\not{d} + M + i\gamma_3(\omega_n + e\tilde{A}_3)] = J_n \det[\not{d} + \rho_n], \tag{15}$$

where

$$J_n = \exp\left(-i\,\frac{e\phi_n}{2\pi}\int d^2x\, \epsilon_{jk}\partial_j A_k\right), \tag{16}$$

with ϵ_{jk} denoting the $(1 + 1)$ Euclidean Levi-Civita symbol.

Recalling the definition of $\Gamma_{\rm odd}$, we see that the second factor in expression (15) does not contribute to it, since it is invariant under $M \rightarrow -M$. As a consequence, the parity-odd piece of the effective action is given in terms of the infinite set of n-dependent Jacobians,

$$\Gamma_{\rm odd} = -\sum_{n=-\infty}^{n=+\infty} \ln J_n = i\,\frac{e}{2\pi}\sum_{n=-\infty}^{n=+\infty}\phi_n \int d^2x\, \epsilon_{jk}\partial_j A_k\,. \tag{17}$$

There only remains to perform the summation over the ϕ_n's, whose sign will obviously depend on the sign of M. Using standard finite-temperature techniques, this series can be exactly evaluated, yielding

$$\Gamma_{\rm odd} = i\,\frac{e}{2\pi}\arctan\left[\tanh\left(\frac{\beta M}{2}\right)\tan\left(\frac{e}{2}\int_0^\beta d\tau\, A_3(\tau)\right)\right] \times \int d^2x\, \epsilon_{jk}\partial_j A_k\,. \tag{18}$$

This is one of the main results in our paper: We have been able to compute the *exact* mass-independent parity-odd piece of the effective action in QED_3 at finite temperature for the restricted set of configurations (3). Several important features of this result should be stressed.

First, this result has the proper zero-temperature limit:

$$\lim_{T\to 0}\Gamma_{\rm odd} = \frac{i}{2}\frac{M}{|M|}S_{\rm CS} = \frac{i}{2}\frac{M}{|M|}\frac{e^2}{4\pi}\int d^3x\, \epsilon_{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha\,. \tag{19}$$

As is well known, in the zero-temperature case the mass-dependent part of the parity breaking is not invariant under large gauge transformations. The quantization of the flux of the magnetic field in the last factor of (18) as $q\frac{2\pi}{e}$ shows that (19) changes by the addition of $inq\pi$ under a large gauge transformation (5). The gauge noninvariance, appearing when n and q are odd, is compensated by the parity anomaly when the result is regularized in a gauge-invariant scheme. Notice also that only in the zero-temperature limit is the result nonanalytic in M.

The same situation occurs in the finite-temperature result (18). A large gauge transformation with odd winding number $n = 2p + 1$ shifts the argument of the tangent in $(2p + 1)\pi$. Although the tangent is not sensitive to such a change, one has to keep track of it by shifting the branch used for arctan definition. This amounts to exactly the same result as in the $T \rightarrow 0$ limit.

Next, we observe that an expansion in powers of e yields the usual perturbative result,

$$\Gamma_{\rm odd} = \frac{i}{2}\tanh\left(\frac{\beta M}{2}\right)S_{\rm CS} + O(e^4), \tag{20}$$

where the coefficient of the Chern-Simons term acquires a smooth temperature dependence. Were we considering only the first nontrivial order in e, we would find a clash between temperature dependence and gauge invariance [16,17]. Now we learn, as it was first stressed in [18] in a $(0 + 1)$-dimensional example and in [19] in a setting similar to ours, that one has to consider the full result in order to analyze gauge invariance. The apparent impossibility of respecting gauge invariance shown by (20) is, in fact, compensated by nonlocal higher order terms in the perturbative expansion.

Finally, we observe that the result (18) is not an extensive quantity in Euclidean time. It is, however, extensive in space, and that is indeed all one expects in finite-temperature field theory. In contrast, the $T = 0$ limit becomes an extensive quantity in space-time, as is expected from zero temperature field theory.

We shall now extend the previous results to the somewhat more general situation of gauge fields satisfying the constraint of A_j, being again time independent, but allowing for a smooth spatial dependence of A_3 besides the previous arbitrary time dependence.

The fermionic determinant we should calculate, after getting rid of the τ dependence of A_3 will have a form analogous to (11) with the only difference being of having an x dependence in ρ_n and ϕ_n. The determinant corresponding to the n mode is again written as in Eq. (13), and we can perform the two-dimensional chiral rotation (14). The x dependence of the phase factor ϕ_n produces, in this case, a different anomalous Jacobian,

$$\det[\not{d} + \rho_n(x) e^{i\gamma_3 \phi_n(x)}] = J_n' \det[\not{d}' + \rho_n(x)], \quad (21)$$

where $\not{d}' = \not{d} - \frac{i}{2} \not{\partial} \phi_n \gamma_3$. This affects the result in two ways. First, as the fermionic operator in the right-hand side depends on the sign of M, there will be a contribution to Γ_{odd} coming from the determinant of $\not{d}' + \rho_n(x)$. Second, the Jacobian is a slightly more involved function of ϕ_n [21],

$$J_n' = \exp\left\{ -i \frac{e}{2\pi} \int d^2x \times \left[\phi_n(x) \epsilon_{jk} \partial_j A_k + \frac{1}{4} \phi_n(x) \Delta \phi_n(x) \right] \right\}. \quad (22)$$

In a first approximation, we shall only take into account the contribution coming from the Jacobian, since the one that follows from the determinant of the Dirac operator is of higher order in a derivative expansion (and we are assuming that the x dependence of $\tilde{A}_3$ is smooth). Moreover, the contribution which is quadratic in ϕ_n is irrelevant to the parity-breaking piece, since it is invariant under the change $M \to -M$. Thus, neglecting the terms containing derivatives of $\tilde{A}_3$, we have for Γ_{odd} a natural generalization of Eq. (18),

$$\Gamma_{\mathrm{odd}} = i \frac{e}{2\pi} \int d^2x \times \arctan\left[\tanh\left(\frac{\beta M}{2} \right) \tan\left(\frac{e}{2} \int_0^\beta d\tau \, A_3(\tau, x) \right) \right] \times \epsilon_{jk} \partial_j A_k(x) . \quad (23)$$

The approximation of neglecting derivatives of $\tilde{A}_3$ is reliable if the condition $|e \partial_j \tilde{A}_3| \ll M^2$ is fulfilled. To end with this example, let us point out that all the remarks we made for the case of a space-independent A_3 apply also to this case.

In conclusion, using particular gauge field configurations, we have computed the mass-dependent parity-violating part of the effective action for $(2 + 1)$ massive fermions at finite temperature obtaining an exact result. Once the standard parity anomaly is taken into account, we have shown that gauge invariance holds even when large gauge field configurations are considered. Our method, which made use of the calculation of the $(1 + 1)$ anomaly, can be also applied to the analysis of the non-Abelian case; details of this case will be given elsewhere.

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