New Class of Random Matrix Ensembles with Multifractal Eigenvectors

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Three recently suggested random matrix ensembles (RME) are linked together to represent a class of RME with multifractal eigenfunction statistics. The generic form of the two-level correlation function for the case of weak and extremely strong multifractality is suggested. [S0031-9007(97)03707-1]

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Random matrix ensembles turn out to be a natural and convenient language to formulate generic statistical properties of energy levels and transmission matrix elements in complex quantum systems. Gaussian random matrix ensembles, first introduced by Wigner and Dyson [1,2] for describing the spectrum of complex nuclei, became very popular in solid state physics as one of the main theoretical tools to study mesoscopic fluctuations [3] in small disordered electronic systems. The success of the random matrix theory (RMT) [2] in mesoscopic physics is due to its extension to the problem of electronic transport based on the Landauer-Büttiker formula [4] and the statistical theory of transmission eigenvalues [5]. Another field where the RMT is exploited very intensively is the problem of the semiclassical approximation in quantum systems whose classical counterpart is chaotic [6]. It turns out [6] that the energy level statistics in true chaotic systems is described by the RMT, in contrast to that in the integrable systems where in most cases it is close to the Poisson statistics.

Apparently the nature of the energy level statistics is related to the structure of eigenfunctions, and more precisely, to the overlapping between different eigenfunctions. This is well illustrated by spectral statistics in a system of noninteracting electrons in a random potential which exhibits the Anderson metal-insulator transition with increasing disorder. At small disorder the electron wave functions are extended and essentially structureless. They overlap very well with each other, resulting in energy level repulsion characteristic of the Wigner-Dyson statistics. On the other hand, in the localized phase electrons are typically localized at different points of the sample, and in the thermodynamic limit where the system size $L \to \infty$ they "do not talk to each other." In this case there is no correlation between eigenvalues, and the energy levels follow the Poisson statistics.

The energy level statistics in the critical region near the Anderson transition turns out to be universal and different from both Wigner-Dyson statistics and the Poisson statistics [7,8]. A remarkable feature of the critical level statistics is that the level number variance $\Sigma_2(\overline{N}) = \langle(\delta N)^2\rangle = \chi\overline{N}$ is asymptotically linear in the mean number of levels $\overline{N} \gg 1$ in the energy window. Such a quasi-Poisson behavior was first predicted in Ref. [9]. Later the existence of the linear term in $\Sigma_2(\overline{N})$ was questioned [8], since for this term to appear the normalization sum rule should be violated. It has been shown recently [10] that the new qualitative feature responsible for the violation of the sum rule and the existence of the finite "level compressibility" χ is the *multifractality* of critical wave functions [11,12].

The notion of multifractality is twofold. The first (and most widely accepted) property of multifractality is related with the space structure of a *single* wave function $\Psi_n(\mathbf{r})$. It is defined through the moments of inverse participation ratio [11]:

$$I_p = \sum_{\mathbf{r}} \langle |\Psi_n(\mathbf{r})|^{2p} \rangle \propto L^{-D_p(p-1)}, \tag{1}$$

where L is the system size, d is the dimensionality of space; $p > 1$ is an integer. The set of exponents $D_p < d$ characterize the fractal dimensionality of the cluster where $|\Psi_n(\mathbf{r})|$ is larger than a certain value that increases with increasing p.

The second, far less appreciated property of multifractality, is related to the overlapping of *different* wave functions with energies E_n and E_m. The main effect of multifractality on spectral statistics is given by the simple overlapping of local densities ($p = 2$). The corresponding fractal dimensionality D_2 is the most important critical exponent. For $|E_n - E_m| \gg \Delta$ ($\Delta = 1/\overline{\rho}L^d$, where $\overline{\rho} = \langle\rho(E)\rangle$ is the mean density of states) the form of the local density correlation function has been suggested and confirmed numerically in Ref. [12]:

$$\langle |\Psi_n(\mathbf{r})|^2 |\Psi_m(\mathbf{r})|^2 \rangle \propto |E_n - E_m|^{-[1-(D_2/d)]}. \tag{2}$$

A remarkable feature of multifractality is that the local density correlation function decreases very slowly with increasing $|E_n - E_m|$ so that two fractal wave functions, however sparse they are, should still overlap strongly [13].

One of the consequences [10] of Eq. (2) is the anomalous Poisson-like term in the level number variance $\Sigma_2(\overline{N})$ which is characterized by the level compressibility χ:

$$\chi = \frac{d - D_2}{2d}. \tag{3}$$

It is immediately seen from Eq. (3) that the critical level compressibility never reaches the Poisson limit $\chi = 1$. For an infinitely sparse fractal $d - D_2 \to d$ is maximal, yet χ is equal to $1/2$ and not to 1. This is because even the infinitely sparse critical fractal eigenfunctions overlap strongly, in contrast to two localized states [10,13].

One may assume that the universal critical level statistics which is described by a set of critical exponents D_p, applies to a wider class of physical systems and it is in fact generic for an intermediate situation between chaos and integrability. An example of such a system has been proposed recently [14]. It turns out that the Coulomb impurity inside an integrable square billiard leads to a drastic reconstruction of eigenstates, however small is the strength of the potential. In such a "Coulomb billiard" all eigenfunctions in the momentum representation exhibit multifractality.

It is therefore natural to look for a RME with multifractal eigenvector and eigenvalue statistics similar to that at the mobility edge in disordered electronic systems. Such a RMT would provide a description of generic features of the critical level statistics.

One such ensemble is suggested in Ref. [15]. It is the Gaussian ensemble of $M \times M$ Hermitian matrices H with independent random entries ($i \geq j$) defined by

$$\langle H_{ij} \rangle = 0, \qquad \langle (H_{ij}^{\mu})^2 \rangle = \beta^{-1}[1 + |i - j|^2/B^2]^{-1}, \tag{4}$$

where H_{ij}^{μ} are real random components ($\mu = 1$ for $i = j$, $\mu = 1, \ldots \beta$ for $i > j$); $\beta = 1, 2, 4$ for Dyson's orthogonal, unitary and symplectic ensembles, and B is a parameter. For $B \gg 1$ this RME can be mapped onto a nonlinear supersymmetric sigma model [15]. The case $B \ll 1$ corresponds to 2D Coulomb billiard considered in Ref. [14]. The presence of multifractality, Eq. (1), and Eq. (3) has been proved [14,15] for this RME. In what follows we will use this RME (RME-I) as a reference point.

There are two more RMEs [16,17] which were suggested recently as possible candidates to describe the critical level statistics. However, their definitions are so drastically different that they were considered as two *alternative* options, albeit the two-level correlation functions (TLCF) $R(\varepsilon, s) = \langle \rho(\varepsilon)\rho(\varepsilon + s) \rangle_c$ in the proper regimes are *identical* for both RMEs. It was first noted in Ref. [18] that since the energy level statistics is a "fingerprint" of the statistics of eigenfunctions, the latter in the corresponding regimes of these two models should also be similar.

The first quantitative link between the predictions of RME equivalent to that studied in Ref. [16] (RME-II) and numerics on the 3D Anderson model at the mobility edge has been done in Ref. [19]. Surprisingly enough it was possible to fit very well the numerics for the critical level spacing distribution $P(s)$ in the 3D Anderson model by a proper choice of only one parameter in RME-II. Moreover, the level compressibility χ in the RME-II for this particular choice of parameter turned out to be very close to that found numerically for the 3D Anderson model.

In this Letter we argue that RME-II and a certain critical regime in RME-III studied in Ref. [17] are equivalent to RME-I and thus possess the multifractality. Altogether they form a new class of RME which describes certain remarkable features of critical level statistics.

We start with the definitions of RMEs studied in Refs. [16,17]. The probability density $P(H)$ for a $M \times M$ random Hamiltonian H from RME-II is given by

$$P_{\mathbf{II}}(H) \propto \exp[-\beta \ \mathrm{Tr} \ V(H)], \tag{5}$$

where the "confinement potential" $V(H)$ grows extremely slowly with H:

$$V(H) = \frac{1}{\gamma} \ln^2 H, \qquad H \gg 1. \tag{6}$$

This is crucial for the universality of the eigenvalue statistics in the limit $M \to \infty$ [20]. For $V(H)$ growing slower than H the full universality is no longer present [21,22], and the eigenvalue statistics may, and does differ from the Wigner-Dyson statistics [16]. Another important feature of Eq. (5) is that the distribution function $P_{\mathbf{II}}(H)$ is invariant under the rotation of basis (unitary invariance):

$$P_{\mathbf{II}}(H) = P_{\mathbf{II}}(UHU^{\dagger}). \tag{7}$$

In contrast to Eq. (5), the distribution function for RME studied in Ref. [17] is Gaussian. However, the unitary invariance is broken by a fixed unitary matrix Ω:

$$P_{\mathbf{III}}(H) \propto e^{-(\beta/2) \ \mathrm{Tr} \ H^2} \, e^{-(\beta/2) b \ \mathrm{Tr}\{[\Omega,H][\Omega,H]^{\dagger}\}}. \tag{8}$$

The properties of this RME depend on the choice of Ω. For the reasons discussed below we consider as RME-III the RME defined by Eq. (8) with $\Omega = \mathrm{diag}(e^{i\theta_j})$, where $\theta_j = 2\pi j/M$. The relevant critical regime for this RME corresponds to the symmetry breaking field b that scales with $M \to \infty$ as $b = h^2 M^2$, where h is a parameter.

Now it is clear why the RMEs given by Eqs. (5) and (8) seem so drastically different. The lack of unitary invariance of $P_{\mathbf{III}}(H)$ means a preferential basis. The existence of such a basis implies a certain structure of eigenfunctions (in this basis) which should lead to spectral statistics different from the Wigner-Dyson one. However, it seems there is no way to get any structure of eigenfunctions in the *unitary-invariant* RME-II. It follows immediately from Eq. (7) that the distribution function $P_{\mathbf{II}}(H)$ depends only on E_n, and the statistics of eigenfunctions in RME-II is trivial and the same as for standard Gaussian ensembles [2]. Then the physical picture that the spectral statistics is related to the statistics of overlapping eigenfunctions seems to leave only one single option: the Wigner-Dyson energy level statistics in RME-II.

Nonetheless, TLCF $R(\varepsilon, s) = \delta(s) + Y_2(\varepsilon, s)$ proves to be *identical* in these RMEs and after unfolding [23] it takes the form [16,17]:

$$Y_2(\varepsilon, s) \approx -\frac{\pi^2\eta^2}{4}\,\frac{\sin^2[\pi s]}{\sinh^2[s\pi^2\eta/2]} \qquad (\beta = 2), \quad (9)$$

where $\eta = \gamma/\pi^2 \ll 1$ or $\eta = h \ll 1$ for RME-II and RME-III, respectively, and $\varepsilon \gg |s|$. Equation (9) coincides with the density correlation function for a free electron gas at a *finite* temperature $\eta\varepsilon_F$ and differs from the RMT result.

The way out from this contradiction is suggested in Ref. [18] where it was conjectured that the unitary invariance is broken in RME-II *spontaneously*. This means that the statistics of eigenfunctions in this ensemble should be calculated *after* an infinitesimal symmetry-breaking term similar to that in Eq. (8) is added. Then the identical TLCF in RME-II and RME-III should be considered as evidence that the proper procedure should result in similar eigenfunction statistics in RME-II and RME-III.

The progress [17] in studying the level statistics in RME-III that lead to Eq. (9) is due to averaging over the unitary group Ω. The level statistics depend on the configuration of eigenvalues $e^{i\theta_j}$ of Ω. The main contribution to the average is made by Ω with the most homogeneous configurations of θ_i, the property known as an eigenvalue repulsion [2,17]. Therefore, one may expect that the spectral statistics obtained *after* such an averaging is close to that corresponding to a *single* unitary matrix Ω with eigenvalues $\Omega_j = \exp[(2\pi i/M)j]$ (RME-III). As a matter of fact for $h \ll 1$ it turns out to be the *same*.

In order to show that we note that in the limit $M \to \infty$ Eq. (8) leads to

$$\langle (H_{ij}^{\mu})^2 \rangle = \frac{1}{\beta}\,\frac{1}{1 + \frac{4\pi^2 b}{M^2}\,|i - j|^2}\,. \qquad (10)$$

If $b/M^2 \to 0$, then we have a standard Gaussian ensemble [1,2] and the Wigner-Dyson statistics. In the opposite case $b/M^2 \to \infty$, we have an ensemble of random diagonal matrices and the Poisson statistics. In the critical case considered here for $b = h^2M^2$, the behavior of $\langle (H_{ij}^{\mu})^2 \rangle$ is the same as in Eq. (4) defining RME-I. We conclude that the $M \to \infty$ limits of RME-I and RME-III coincide. Then TLCF for RME-III and RME-I must be identical.

Fortunately, the latter can be calculated directly. TLCF can be expressed [24] in terms of the spectral determinant $P(s)$ as follows:

$$Y_2(s) = -\frac{1}{2\pi^2 s^2} - \frac{1}{4\pi^2}\,\frac{d^2}{ds^2}\ln P(s) + \frac{\cos(2\pi s)}{2\pi^2 s^2}\,P(s)\,, \qquad (11)$$

where

$$P(s) = \prod_{n\neq 0}\left(1 + \frac{s^2}{\epsilon_n^2}\right)^{-1}, \qquad (12)$$

and ϵ_n is a spectrum of the quasidiffusion modes. The latter can be found from the mapping [15] onto the nonlinear σ model ($B \gg 1$) as follows: $\epsilon_n = 4B|n|$, where for the periodic boundary conditions $n = \pm 1, \pm 2, \ldots$. Making use of the identity $x^{-1}\sinh x = \prod_{n=1}^{\infty}(1 + x^2/\pi^2 n^2)$ we immediately arrive at Eq. (9) with $\eta = 1/(2\pi B)$. Using the results of Ref. [15] one can express the multifractality exponent D_2 in terms of $B \gg 1$. For $\beta = 2$ it appears to be $D_2 = 1 - 1/(2\pi B)$ which helps to identify the parameter η in Eq. (9) as $\eta = 1 - D_2$. Thus all three ensembles share the same TLCF, Eq. (9) which is generic to RME with weak multifractality $\eta \ll 1$.

The level compressibility χ in Eq. (3) is obtained by the integral of the TLCF [8,10]:

$$\chi = 1 + \int_{-\infty}^{+\infty} Y_2(\varepsilon, s)\, ds. \qquad (13)$$

Using Eq. (9) one can calculate the level compressibility $\chi = \eta/2 + [1 - \coth(2/\eta)]$ in the limit of weak multifractality $\eta \ll 1$. Neglecting the exponentially small terms, we observe that Eq. (3) (with $d = 1$) is fulfilled.

Note that both the linear level number variance $\Sigma_2(\overline{N})$ with $\chi \neq 1$ and TLCF of the form Eq. (9) are *not* the trivial consequences of the basis preference. A good counterexample is a Gaussian RME with the variance of the fluctuating diagonal components different from that of the off-diagonal ones. Their ratio $\mu = M^2/\lambda^2$ sets the new energy scale $\lambda \gg 1$ in the problem, such that for $s \gg \lambda$ spectral correlations deviate from the Wigner-Dyson form. However, the recent analytical results [25,26] show that this deviation is qualitatively different from that described by Eq. (9) for $s \gg 1/\eta$. Albeit the oscillations in $Y_2(s)$ die out for $s \gg \lambda$, there is still left a constant tail $Y_2(s) = 1/\pi^2\lambda^2$ that extends up to $s = \lambda^2$. Therefore the level number variance $\Sigma_2(\overline{N}) = (\overline{N}/\pi\lambda)^2$ for $N \ll \lambda^2$ and $\Sigma_2(\overline{N}) = \overline{N}$ for $N \gg \lambda^2$.

With increasing γ and h or decreasing B the fractal dimensionality D_2 decreases and Eq. (9) is no longer valid. It is reasonable to assume that in the limit $h, \gamma \to \infty$ or $B \to 0$ the fractal eigenvector becomes infinitely sparse, $D_p \to 0$. For RME-I this is, indeed, the case [14]. In this limit Eq. (3) predicts $\chi = 1/2$. Let us check this prediction using an exact form of TLCF given in Ref. [16].

First of all we note that even after unfolding, the TLCF $R(\varepsilon, s)$ in RME-II is not invariant under a shift in ε. In the limit $\gamma \to \infty$ the TLCF has the same form in the orthogonal, unitary, and symplectic ensembles [19,22]:

$$Y_2(\varepsilon, s) = -\theta(1/4 - |s - \delta|)\,, \qquad (14)$$

where $-1 < 4\delta < 1$ is a deviation of $4\varepsilon \gg 1$ from the nearest odd integer, and $\theta(x)$ is a step function.

The lack of translational invariance is a peculiarity of the particular RME-II. Only the TLCF *smoothened* by averaging over δ can be physically meaningful. So we arrive at the TLCF of the triangle form:

$$Y_2(s) = \begin{cases} 2|s| - 1, & |s| < 1/2, \\ 0 & \text{otherwise}\,. \end{cases} \qquad (\beta = 1, 2, 4) \quad (15)$$

It is remarkable that after the substitution into Eq. (13) of either Eqs. (14) or (15) we get the predicted value $\chi = 1/2$. One may thus expect [27] that the triangle form of the TLCF is generic for all critical states in the limit of an infinitely sparse fractal $D_2 \to 0$ [28].

Let us discuss the applicability of Eqs. (9) and (15) to the description of spectral statistics at the Anderson transition. The point is that for RMEs considered here all states are critical. For disordered electron systems there is a mobility edge E^*, and the states are nearly critical only in its vicinity where $\xi(E) \propto |(E - E^*)/E^*|^{-\nu} > L$. Therefore the RMEs considered here correspond formally to $\xi(E) = \infty$ and thus $\nu = \infty$. Indeed, it has been shown [8] that for finite ν the critical TLCF should have a power-law tail $R(s) = -c_d|s|^{-[1+(1/\nu d)]}$, where $c_d \sim 1/(\pi^2 \nu d)$. It vanishes in the limit $\nu \to \infty$ in agreement with the exponential decay of TCLF given by Eq. (9). However, even for the realistic case $\nu \sim 1$ the power-law tail is small due to the additional factor $\pi^2 d$.

In conclusion, we link together three different random matrix ensembles suggested recently. Since in one of them the eigenfunction statistics is known to be multifractal, we argue that all three RME belong to the same universality class with the multifractal eigenfunction statistics. By combining known solutions for all three ensembles we suggest the form of the two-level correlation function in the region of weak and extremely strong multifractality.

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[28] Up to a rescaling, Eq. (15) has the same form as $K(t) = \int Y_2(s)\, e^{ist} ds$ for $\beta = 2$ in the RMT limit $\gamma \to 0$. Thus there is an interesting *duality* $Y_2(s)|_{\tilde{\eta}=1} \sim K(t)|_{\tilde{\eta}=0}$.