Ultrametric Structure of Multiscale Energy Correlations in Turbulent Models

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The ultrametric structure of the energy cascade process in a dynamical model of turbulence is studied. The tree model we use can be viewed as an approximated one-dimensional truncation of the wavelets-resolved Navier-Stokes dynamics. Varying the tree connectiveness, the appearance of a scaling transition in the two-point moments of energy dissipation is detected, in agreement with experimental turbulent data. [S0031-9007(97)03998-7]

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Spatiotemporal intermittency is the most intriguing aspect of a fully developed three-dimensional turbulent flow.

Experiments [1] show that the energy dissipation defines a multifractal measure on the fluid volume. The multifractal measure is characterized by the scaling properties of the coarse-grained energy dissipation on a box at scale r, ε_r, namely, $\langle[\varepsilon_r(x)]^p\rangle \sim r^{\tau(p)}$, where $\langle\cdot\rangle$ means averaging over all boxes of size r and centered in x in which the volume can be partitioned. The measured $\tau(p)$ exponents show a clear intermittent behavior, i.e., a nonlinear dependency on the order of the moment p.

The simplest way to explain phenomenologically the presence of intermittent deviations consists in describing the energy transfer mechanism in terms of fragmentation stochastic processes. In these models (see [2] for a recent proposal), one introduces a set of eddies leaving on a dyadic structure and connected through a random multiplicative process.

Let us remark that all stochastic fragmentation models so far proposed lack any direct linking with the original Navier-Stokes (NS) equations. Dynamical deterministic models on hierarchical structures are therefore invoked for improving our understanding of the energy transfer mechanism.

In this paper, we study dynamical models which fill the gap between purely stochastic fragmentation models and the original NS dynamics. In particular, we consider a dynamical model on one-space and one-time dimensions [3]. One can look at this model as an approximation of the original NS equations in a wavelets basis [4–6].

In order to specify the model one has to select the set of interactions connecting eddies at different scales and at different spatial locations. By changing the interactions set, one changes the scale organization of energy structures: in order to study it, new tools are required [7], which characterize intermittency more completely than the multifractal spectrum $\tau(p)$ alone. An obvious generalization of the single-point statistics is to inquire about the scaling of two-point moments. In practice, introducing the mixed moments,

$$\langle[\varepsilon_r(x)]^q[\varepsilon_l(x+s)]^p\rangle, \tag{1}$$

one can study correlations between different scales by changing r and l and/or correlations between different points in the fluid volume by changing s.

Different interactions among nodes of the hierarchical dynamical structure can lead to very different prediction for the scaling behavior of (1).

An ultrametric organization of the tree can be detected by looking for a phase transition in the set of scaling exponents characterizing correlations (1). An ultrametric space (see [8] for a review) is defined as a metric space endowed with a distance $d(a,b)$ satisfying the inequality $d(a,c) \le \max(d(a,b), d(b,c))$. It is simple to see that a multiplicative cascade process is characterized by an ultrametric organization. Indeed, let us consider two eddies of scale size $l_n = 2^{-n}L_0$ (L_0 being the integral scale), and let us assume that the two eddies have a smallest common "ancestor" (in terms of the cascade process) of scale l_m; that is, l_m is the size of the smallest eddy containing the two eddies at scale l_n. Then, by defining the distance as $\log_2(l_m/l_n)$, one can easily see that the ultrametric inequality is verified.

In [7] the analysis of the two-point observables (1) performed on experimental turbulent data gave a first support for an ultrametric organization of the main triadic interactions in Navier-Stokes equations.

In the following, we are going to analyze the same kind of observables measured on a direct numerical integration of a dyadic-tree model for turbulent energy cascade. In particular, by changing the set of interacting triads we want to disentangle the basic symmetries behind the transition observed in real turbulent data.

Let us turn to a brief review of the model (for a comprehensive description see [3]). The tree model can be viewed as an extension of shell models, which can be seen as a severe truncation of the NS equations (see [9] for a general introduction). The most popular shell model is

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the Gledzer-Ohkitani-Yamada (GOY) model ([10]–[15]). Recently, a new class of shell models based upon the helical decomposition of NS equations [16] has been suggested [17] and studied [18,19]. In these models, one or few complex variables u_n represent an entire *shell* of wave numbers k such that $k_n < k < k_{n+1}$, with $k_n = 2^n$. Shell models can be thought of as field problems in zero spatial dimension ($d = 0$). In order to include also some real space dynamics we need to transform the *chain* model into a *tree* model with $d = 1$. This is achieved by letting grow the number of degrees of freedom with the shell index n as 2^n. The tree model can be regarded as describing the evolution of the coefficients of an orthonormal wavelets expansion of a one-dimensional projection of the velocity field. In the tree model, we use the notation $u_{n,j}^{\pm}$ to indicate a complex variable having positive or negative defined helicity and living on scale k_n and spatial position labeled by the index j. For a given shell n, the index j can vary from 1 to 2^{n-1}.

We report here the structure of the tree model dynamical equations (for more details, see [3]),

$$\dot{u}_{n,j}^{+} = ik_n \sum_{n_1,n_2,j_1,j_2} [a_{n_1,n_2,j_1,j_2} u_{n_1,j_1}^{s_1} u_{n_2,j_2}^{s_2}]^* - \nu k_n^2 u_{n,j}^{+} + \delta_{n,n_0} F^{+}. \tag{2}$$

Here, $n = 1, \ldots, N$, where N is the total number of shells, ν is the viscosity, F^+ the external forcing acting on the large-scale shell $n_0 = 1$, and a_{n_1,n_2,j_1,j_2} are parameters, which are determined by imposing conservation of energy and helicity in the inviscid and unforced limit. The s_1 and s_2 indices are the helicity signs ($\pm$) of interacting modes. The same equations hold, with all helicities reversed, for $u_{n,j}^{-}$.

In restricting the possible choices of the nonlinear terms, we can phenomenologically require a certain degree of locality for interactions among variables at different scales and at different spatial positions. Regarding scale numbers n_1 and n_2, in our Eq. (2) each variable $u_{n,j}^{+}$ is allowed to interact only with nearest and next-to-nearest levels: indeed, n_1 and n_2 can vary only from $(n - 2)$ to $(n + 2)$. Regarding space numbers j_1 and j_2, we define two different models having different topological structures of the dynamical interactions. The first model, hereafter called model A, is a model which has an *ultrametric* structure. That is, each eddy is allowed to interact only with bigger eddies which spatially contain it and with smaller eddies spatially contained in it. This model is the natural dynamical representative of all stochastic fragmentation models which phenomenologically reproduce single-point intermittent exponents. The second model, hereafter called model B, has an enlarged interactions set containing also horizontal couplings, which allow eddies covering different spatial regions to interact with each other. In Figs. 1 and 2, we pictorially show the set of interactions defining models A and B.

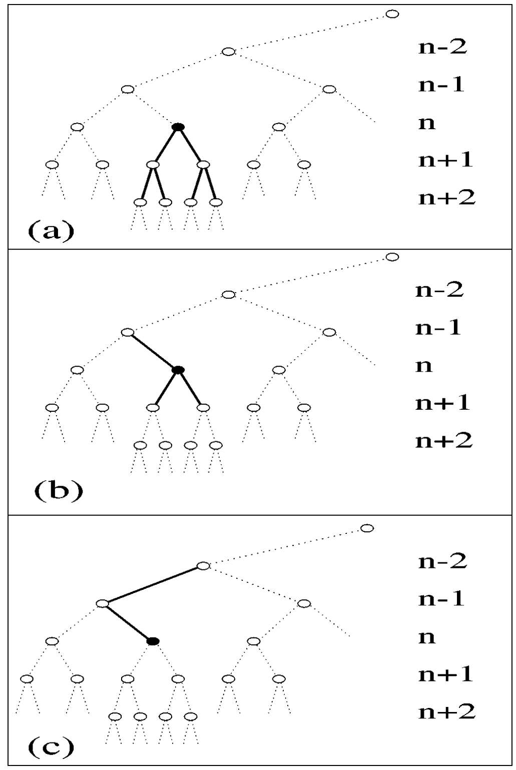

FIG. 1. Pictorial representation of interactions in model A: in this case the nonlinear terms in (2) result from the sum of parts (a), (b), and (c) in the figure.

The single-point statistical properties of the tree model have been studied in [3]: in both cases A and B, the system turned out to have an intermittent energy transfer qualitatively similar to what one can find in the original NS equations. The treelike structure imposed on the velocity fluctuations does not necessarily imply that the energy dissipation can be described in terms of fragmentation processes. In order to test the scale organization of the energy structures, ultrametric-sensitive observables should be studied.

In order to compare our system with previous findings [7], we shall detect possible ultrametric structure in the energy cascade of the tree model looking at two-point statistical quantities. All parameter settings and numerical methods are as in [3]. In particular, we consider a total number of levels $N = 16$: the total number of sites forming the tree is then $N_T = 2^N - 1 = 65\,535$, each one described in terms of two complex variables. Numerical simulations needed state-of-the-art multiprocessor computers.

The fields we focus on are the coarse-grained energy dissipation densities, here denoted as $\epsilon_n(j)$, obtained as averages over spatial regions $\Lambda_j(n)$ of length 2^{-n}. We consider the mixed moments (1) with $r = l$ and $p = -q$, which in our notation become

$$\langle \varepsilon_n^q(j) \varepsilon_n^{-q}(j + s) \rangle \equiv C_n(q,s). \tag{3}$$

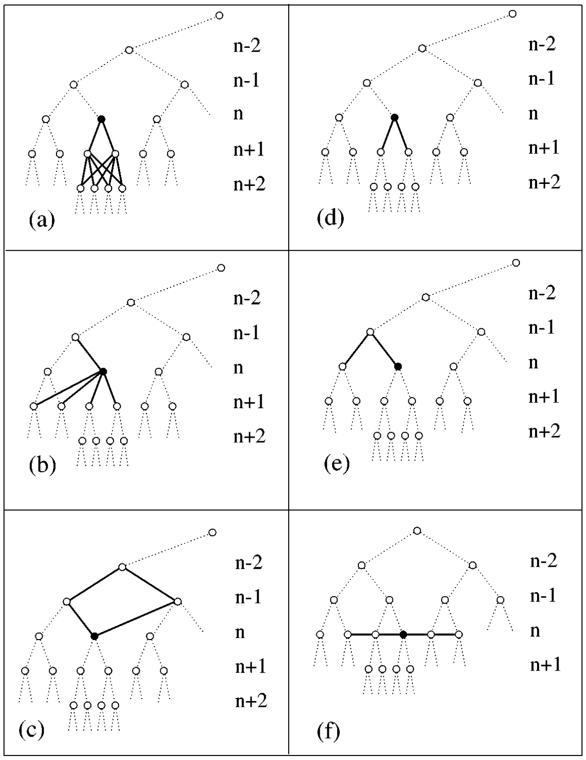

FIG. 2. Pictorial representation of interactions in model B: in this case the nonlinear terms in (2) result from the sum of parts (a), (b), (c), (d), (e), and (f) in the figure.

The behavior of this quantity for intermittent ultrametric measures resulting from random multiplicative processes has already been analyzed in the framework of the two-point multifractal formalism [7,20]. In this case, the average can be properly decomposed and a general result can be obtained for its dependence on the spatial distance s between the two points,

$$C_n(q,s) \sim s^{\min[-\tau(-q)-\tau(q),1]} \equiv s^{\Phi(q)}. \tag{4}$$

This expression implies that for some moment q, a sharp transition occurs in the derivative of the scaling exponent $\Phi(q)$. This scaling transition is the analog of a phase transition in the thermodynamic interpretation of multifractals [21]. The behavior of (3) is dominated by pairs of points at which the dissipation is very large at one point and very low at the other. This constitutes the subset of points that are likely to be independent from each other and lie on the boundary of their bigger predecessors. Indeed, it must be recalled that in an ultrametric structure nearby (in space) eddies could lie on the boundaries of much bigger ones, then having an effective large ultrametric distance.

The two spatial scales of interest are the coarse-graining scale $l_n = 2^{-n}$ and the offset scale $l_s = l_n s$: they should be such that $\eta \ll l_n \ll l_s \ll \Lambda_T$, where η and Λ_T are the Kolmogorov and integral scales, respectively. For this reason, we fixed $n = 11$, in order to consider the largest inertial scale in our tree structure, and we let s assume the exponentially spaced values $s = 2^m$, with $m = 1, 2, \ldots, n-2$. In order to test the presence of a scaling transition, the mixed moments (3) have been computed for $0.5 < q < 4$.

Figure 3 shows the mixed moments (4) as a function of s in log-log coordinates, for increasing values of q and for the two versions A and B of the tree model. The $\Phi(q)$ exponents have been calculated by linear fit in the inertial range region: they are reported in Fig. 4, where they are compared with the curve $[-\tau(q) - \tau(-q)]$, obtained using the single-point moments exponents and corresponding to the predicted form of $\Phi(q)$ if the scaling transition were absent. In the case A, the data support a sharp transition in the derivative of $\Phi(q)$ at $q \sim 1.5$, with a much slower variation of $\Phi(q)$ for $q > 1.5$. This transition is absent in case B.

We thus conclude that version A gives support for a scaling transition in the mixed moments of coarse-grained dissipation. This result agrees with the experimental behavior found in [7] (see Fig. 17 of this reference) using data measured in a turbulent wake. The physical picture implied by this scaling transition is that of uncorrelated small eddies that come close together even sharing no common history during the energy cascade.

Let us summarize our results.

A dynamical model in one spatial dimension originating from a wavelets-like decomposition of a one-dimensional cut of a turbulent velocity field has been studied. We found that a scaling transition appears as soon as the tree has a pure ultrametric dynamical structure.

The fact that decreasing the number of triad interactions one can reproduce the real data scaling transition observed in [7] may seem in contrast with the observation that in the original Navier-Stokes equations all possible interactions are switched on. This contradiction is only apparent: divergenceless character of the original NS field, added to complex phase-coherence effects, can very easily introduce different dynamical weights in the possible triad interactions, leading to a situation where only a few of them govern the global dynamical evolution. For example, Grossmann and co-workers showed [5,22], by performing suitable truncation of NS equations, that intermittency depends on the typical degree of locality in

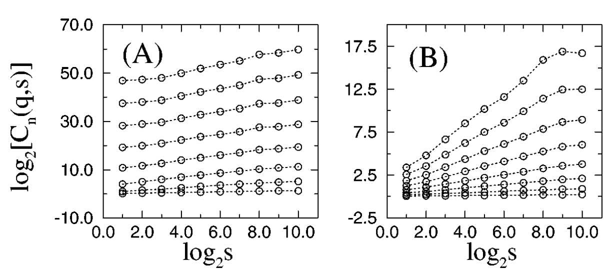

FIG. 3. Log-log plot of mixed moments of order $q = 0.5, 1, 1.5, \ldots, 4$ (from bottom to top) against the distance s, for models A (left) and B (right).

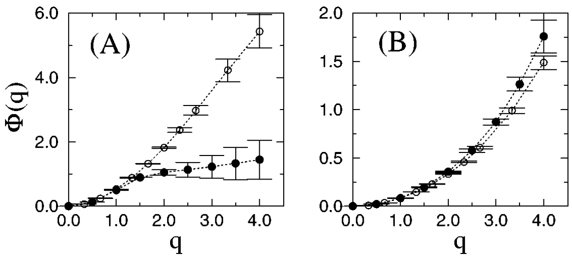

FIG. 4. The exponents $\Phi(q)$ (solid circles) as a function of q for models A (left) and B (right). For comparison, the values of the function $[-\tau(q) - \tau(-q)]$ are also reported (open circles).

Fourier space of the survived triad interactions; very similar results have also been found in shell models at varying the intershell ratio λ [19].

Recently, some theoretical studies and experimental analysis have been done on multipoint multiscale velocity correlation functions in turbulent flows [23]. Our dynamical investigation suggests that, in the presence of a strong ultrametric structure, correlations among velocity fluctuations at different scales should depend on their ultrametric distance rather than on the separation length only, as predicted in [23].

These and similar studies performed on such kinds of models can improve our understanding of basic mechanisms underlying turbulent cascade. For instance, it is important to recognize those interactions which are more effective in the energy transfer mechanism when constructing eddy-viscosity models and in simulating small scale statistics by some closure approach.

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