The Light Quark Masses from Lattice Gauge Theory

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We study the masses of the light quarks with lattice QCD. Most of the dependence on the lattice spacing a, observed previously with Wilson fermions, is removed by an $O(a)$ corrected action. In the quenched approximation, we obtain a strange quark $\overline{MS}$ mass $\overline{m}_s(2\ \mathrm{GeV}) = 95(16)$ MeV, and an average of the up and down quark masses $\overline{m}_l(2\ \mathrm{GeV}) = 3.6(6)$ MeV. Correcting for quenching, the masses likely are 20% to 40% smaller: $54 < \overline{m}_s(2\ \mathrm{GeV}) < 92$ MeV and $2.1 < \overline{m}_l(2\ \mathrm{GeV}) < 3.5$ MeV. We argue that most lattice determinations are consistent with these low values, which are outside the range conventionally given. [S0031-9007(97)03896-9]

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Among the most important applications of lattice gauge theory to particle physics are the calculations required to determine the fundamental parameters of the quark sector of the standard model. One of the most important of these is the overall scale of the light quark masses. It is one of the least well known of the fundamental parameters of the standard model. (Estimates for the strange quark mass range from 100 to 300 MeV for the $\overline{MS}$ masses renormalized at a "high" energy scale, 1 GeV, and for the average light quark mass from 3.5 to 11.5 MeV [1].) It is also one for which lattice methods are almost uniquely reliable, unlike quark mass ratios or the strong coupling constant α_s, for which other powerful methods exist. Values for quark masses have been obtained since almost the beginning of lattice phenomenology [2,3]. However, improved understanding of perturbation theory and finite lattice spacing errors has been required to make sense of the various lattice determinations, which initially ranged over a factor of 3.

Lattice determinations of standard model parameters consist of two pieces. Calculations of experimentally measurable quantities such as hadron masses are used to fix the bare coupling and quark masses in the lattice Lagrangian. Short distance calculations are used to relate the bare parameters in the lattice theory to renormalized, running couplings and masses, such as those of the $\overline{MS}$ scheme.

Quark masses are best obtained in lattice calculations by matching pseudoscalar meson masses to experiment. These are among the easiest lattice calculations, with small statistical and finite volume errors. Experimental uncertainties are also negligible. Uncertainties are dominated by truncation of perturbation theory and discretization errors, and by errors arising from the omission of light quark loops (the "quenched" approximation).

The calculations relating the parameters in various regulators may be performed by equating short distance quantities. It is desirable to do the lattice part of such calculations as nonperturbatively as possible, to test for nonperturbative short distance effects and possible poor convergence of perturbation theory. But nonperturbative short distance analyses for quark masses are currently not far advanced.

Perturbative relations between the lattice bare mass m_0 and the $\overline{MS}$ mass $\overline{m}$ may be obtained by demanding that on-shell Green functions calculated with both regulators be equal. Analogous perturbative expressions for the renormalization of α_s were initially rendered almost useless by sick behavior in the lattice perturbation series. In Ref. [4] it was shown that such behavior could be understood and mostly eliminated by a mean field theory resummation of large "tadpole" graphs.

To reduce the effects of such graphs further, the relation between $\overline{m}$ and m_0 may be rewritten with a mean field improved mass $\tilde{m}$,

$$\overline{m}(\mu) = \tilde{m}[1 + \alpha_s \gamma_0 (\ln \tilde{C}_m - \ln a\mu) + \cdots], \quad (1)$$

where $\gamma_0 = 2/\pi$ is the leading anomalous dimension, and $\ln \tilde{C}_m$ is the result of a one loop calculation. Here $\tilde{m} = \ln(1 + 1/2\tilde{\kappa} - 1/2\tilde{\kappa}_c)$, with the mean field improved hopping parameter $\tilde{\kappa} \equiv \kappa u_0$, and $u_0 \equiv \langle U_P \rangle^{1/4}$. The nonperturbative value of the plaquette expectation value $\langle U_P \rangle$ incorporates an estimate of higher order tadpole graphs into $\tilde{m}$. The one loop term $\ln \tilde{C}_m$ is adjusted to remove the one loop part of this expression $u_0 = 1 - \pi\alpha_s/3$. The ellipsis denotes higher orders in α_s^2 and in a.

In Fig. 1 we show a compilation of previous results [2]. Quenched results obtained with staggered fermions are almost cut-off independent for lattice spacings less than 1 GeV^{-1}. However, for staggered fermions the constant in Eq. (1) is $\tilde{C}_m = 132.9$ [5]. The one-loop relation is thus of doubtful reliability: The correction is 50%–100%, most of which is unexplained by mean field theory.

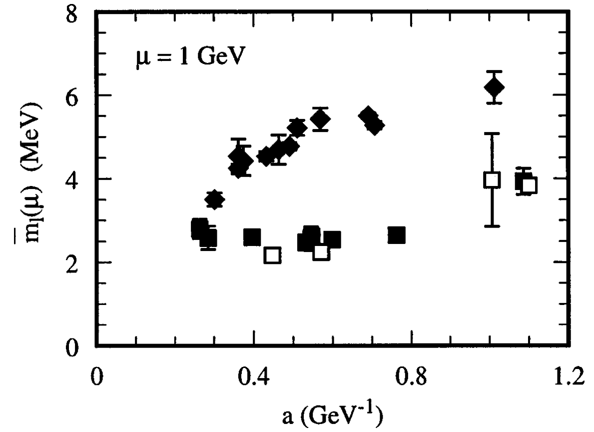

FIG. 1. Previous lattice results for the $\overline{MS}$ masses of the light quarks, renormalized at 1 GeV, with the lattice spacing set by the ρ mass. Lattice spacing dependence is large for quenched Wilson fermions (diamonds) and small for quenched staggered fermions (filled squares). Results from two-flavor staggered fermion QCD (open squares) lie below those from quenched approximation staggered fermions. Data from Ref. [2].

For Wilson fermions $\tilde{C}_m = 1.67$ [5], so the perturbation series behaves well (to first order in α_s). However, the results for the Wilson action show strong cut-off dependence. They lie far above the results for staggered fermions and fall as the lattice spacing is reduced. The Wilson action contains an error of $O(a)$, which is absent for staggered fermions. After extrapolating in a, the result is much closer to the results of staggered fermions. (See, for example, Ref. [6].) However, further sources of cut-off dependence are an unknown combination of $O(\alpha_s^2)$, $O(\alpha_s a)$, $O(a^2)$, etc. Without a full theory of their functional form one cannot extrapolate confidently.

One should therefore try to remove the dominant $O(a)$ error from the Wilson action. A convenient action for doing so [7] incorporates an extra dimension 5 term $c_{\mathrm{SW}}\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$, the so-called "clover" term, whose coefficient can be adjusted to remove the $O(a)$ error. At tree level $c_{\mathrm{SW}} = 1$, but the one-loop correction to the coefficient of the clover term is large [8], as suggested by mean field theory [4]. It is a three-tadpole correction and can be approximated by $c_{\mathrm{SW}} \approx u_0^{-3}$. For the improved action, $\tilde{C}_m = 4.72$ [9], so Eq. (1) is still well behaved (to one loop).

We use this action to determine the overall scale of the light quark masses. We fit for the coefficient of m_l in the expression $M_\pi^2 = Bm_l + \ldots$. (We do not see deviations from this leading-order equation, see below.) The slope B is the main result of our calculation, although we present results as the quark masses $\overline{m}_l$ and $\overline{m}_s$ for comparison with Ref. [1]. Our lattice spacings range from 1.26 GeV^{-1} (where perturbative uncertainties are nearly 50%), down to 0.39 GeV^{-1} (where perturbation theory appears well behaved). We have performed the calculation at the largest lattice spacing to investigate its behavior where perturbation theory is beginning to break down, but we omit it from the analysis leading to our final results. The lattice spacings have been obtained from the $1P$-$1S$ splitting of the charmonium system, $\Delta M \equiv M_{h_c} - (3M_{J/\psi} + M_{\eta_c})/4$, for which the uncertainties of lattice calculations are particularly small and easy to understand. Numerical uncertainties in our results for the quark masses thus arise from a combination of uncertainties in the charmonium and pion calculations. See Ref. [10] for more details.

We use improved lattice perturbation theory to convert to the $\overline{MS}$ mass at renormalization scale $\mu = 2$ GeV and ΔM to determine the lattice spacing, whereas previous work typically used bare perturbation theory at scale $\mu = 1$ GeV and the ρ meson mass to determine the lattice spacing. Although renormalization at 1 GeV is conventional in nonlattice results, the low scale induces additional perturbative uncertainty, which is not present in the underlying lattice results.

Charmonium calculations are discussed in Ref. [11]. Some details and results of our pion calculations are given in Table I. We calculated the pion mass from correlated fits of 2×2 correlation functions (using two operators and fitting two states), with statistical errors from 1000 bootstrap samples. On the smaller lattices we checked for contamination from excited states by comparing with 1×1 and 3×3 fits. Detailed descriptions of our numerical methods are in preparation [12].

In Table I and Fig. 2 we give our results for the light quark masses in the quenched approximation. We apply Eq. (1) at the scales $\mu = 1/a$ and π/a and then run to 2 GeV. The errors shown are statistical only. The diamonds are our results for unimproved Wilson fermions. They are consistent with the existing work (diamonds in Fig. 1). The triangles are our results for the mean-field-improved clover action. Most, but not all, of the cut-off dependence has been removed.

Remaining sources of cut-off dependence could include large α_s^2 corrections to the mass relation, Eq. (1), further corrections to the clover coefficient in the pion numerical calculations, and $O(a^2)$ corrections to ΔM. Leading $O(a)$ corrections are expected to be negligible for ΔM, but quark momenta are larger in charmonium than in pions, and we estimate $O(a^2p^2)$ corrections to ΔM to range from 4%–20% on our three finest lattice spacings. The one-loop result [8] for the coefficient c_{SW} agrees with the

TABLE I. Our results for $\overline{m}_l$(2 GeV), the average of the u and d quark masses, renormalized at 2 GeV.

β	5.5	5.7	5.9	6.1
a (GeV^{-1})	1.26	0.86	0.57	0.39
volume	$8^3 \times 16$	$12^3 \times 24$	$16^3 \times 32$	$24^3 \times 48$
$\overline{m}_l$ ($c = 0$)	6.31(26)	5.93(17)	4.88(18)	4.62(22)
$\overline{m}_l$ (improved)	4.75(19)	4.41(12)	3.90(13)	3.84(18)
c_{SW}	1.69	1.57	1.50	1.40

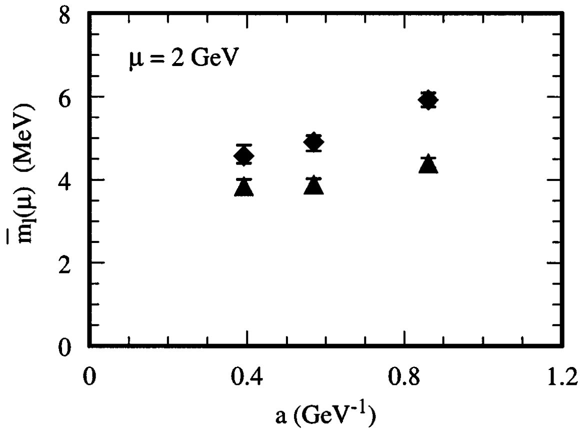

FIG. 2. Our results for the masses of the light quarks. Most of the lattice spacing dependence of unimproved Wilson fermions (diamonds) is removed by the use of an $O(a)$ corrected action (triangles) with a tadpole improved c_{SW}. The lattice spacing is set by the charmonium $1P$-$1S$ splitting ΔM. Errors are statistical only.

mean field estimate, but a nonperturbative determination favors an even larger correction [13]. Purely perturbative errors in the relation between the lattice and $\overline{MS}$ masses should be of order $\alpha_s^2 \sim 5\%$ at our finest lattice spacing. Other smaller uncertainties include finite volume effects, which should be a couple of percent or less, and statistical errors, which are 4% and arise mostly from the lattice spacing from the charmonium system.

We examined the pseudoscalar meson mass squared as a function of the quark mass. It should be linear plus small corrections in the small quark mass limit. Our numerical data are for quark masses in the range $0.4m_s$ to m_s. In this range, we find no statistically significant evidence for quadratic terms in M_π^2 vs m_l, much less the large quadratic terms that have been postulated to make $m_u = 0$ consistent with experiment. Therefore, our results for the ratio of the strange to light quark masses agree with lowest order chiral perturbation theory: $(m_s + m_l)/(2m_l) \approx M_{K^0}^2/M_{\pi^0}^2 \approx 13.6$.

Despite significant reduction, the small remaining cut-off dependence produces the least reliably understood uncertainty in the quenched approximation. Pending further understanding of this error, we take our result at the smallest lattice spacing as the top of our lattice spacing error bar. We take a linearly extrapolated result (the lower of two plausible extrapolation methods) through the three finest lattice spacings as the bottom. This gives a range of 0.8 MeV for the cut-off dependence uncertainty. The perturbative uncertainty was added to this range by applying the perturbative correction using α_V (as defined in Ref. [4]) renormalized at the two scales $1/a$ and π/a to the results of the two extrapolations. The outermost of the four results were taken as the errors bars, increasing the errors by 5%, or about α^2. This gives our continuum limit, quenched approximation result:

$$\begin{aligned} \overline{m}_l(2\ \mathrm{GeV}) &= 3.6(6)\ \mathrm{MeV}, \\ \overline{m}_s(2\ \mathrm{GeV}) &= 95(16)\ \mathrm{MeV}. \end{aligned} \tag{2}$$

The perturbative and cut-off dependence uncertainties are added linearly, since they are related. All others are added in quadrature to the total error.

We now consider the most poorly understood source of uncertainty, the quenched approximation. Without light flavors in loops, QCD couplings run incorrectly. For example, α_s runs too fast [11]. To leading logarithmic accuracy, $\alpha_s(\pi/a)$ is too small by a factor of about $\beta_0^{(3)}/\beta_0^{(0)}$, where $\beta_0^{(0)}$ and $\beta_0^{(3)}$ are the leading quenched and unquenched β functions, respectively. The running of the quark mass is proportional to α_s, in the perturbative region. It runs too slowly in the quenched approximation, leading to too large a quark mass at short distances. In Ref. [14], the ratio of quenched and unquenched quark masses arising from the perturbative region was estimated, to leading logarithmic accuracy, to be

$$\frac{m(\pi/a)|_{\mathrm{qu}}}{m(\pi/a)|_{\mathrm{unqu}}} \approx \alpha_s(\pi/a)^{(\gamma_0/\beta_0^{(0)} - \gamma_0/\beta_0^{(3)})/2} \approx 1.2, \tag{3}$$

for $\alpha_s(\pi/a) \approx 1/6$ to $1/8$. There is, of course, a further unknown contribution from the nonperturbative region. Still, a reduction of tens of per cent due to light quark loops in the perturbative region is not unexpected.

Some unquenched staggered results summarized in Ref. [2] are shown in Fig. 1. (Unquenched Wilson fermion calculations appear to be more difficult to perform and harder to interpret.) The unquenched results lie below the quenched results, roughly as expected. Although the unquenched calculations are not yet as solid as the quenched calculations, we take them seriously enough for estimating the gross effect of quenching. We argued above that quenched staggered quark masses look good, except that the large $\ln \tilde{C}_m$ points to poor convergence of perturbation theory. However, the ratio of quenched to unquenched quark masses is useful, because the perturbative factor cancels. To minimize effects due to differences in analysis methods, we estimate the ratio from the results of a single group [15,16], at similar volumes and lattice spacings (about 0.4 GeV^{-1}), finding

$$\frac{\overline{m}_l(1.0\ \mathrm{GeV})_{n_f=0}}{\overline{m}_l(1.0\ \mathrm{GeV})_{n_f=2}} \approx \frac{2.61(9)}{2.16(10)} = 1.21(7). \tag{4}$$

Since there are three flavors of light quarks, not two, we use this ratio as a lower bound on the actual ratio and its square (i.e., four light quarks) as an upper bound.

Combining our quenched result, Eq. (2), with the correction ratio suggested by Eq. (4), we obtain the unquenched results

$$\overline{m}_s(2\ \mathrm{GeV})\ \text{in the range}\ 54\text{–}92\ \mathrm{MeV},$$

$$\overline{m}_l(2\ \mathrm{GeV})\ \text{in the range}\ 2.1\text{–}3.5\ \mathrm{MeV}.$$

Running down to 1 GeV, where conventional mass estimates are often quoted, the estimates are raised by about 10%, to $\overline{m}_s(1\ \text{GeV})$ in the range 59–101 MeV, and $\overline{m}_l(1\ \text{GeV})$ in the range 2.3–3.9 MeV.

Another determination [17] of the quenched strange quark mass obtains $\overline{m}_s(2\ \text{GeV}) = 128(18)$ MeV. This is an average of results with the Wilson action and the tree-level clover action. Reference [17] makes no attempt to remove the remaining lattice spacing dependence or to estimate the effects of the quenched approximation. Most of the discrepancy with our quenched results arises from the treatment of cut-off effects. Again, we try to reduce cut-off effects with much larger clover coefficients, and by extrapolating away remaining effects even so.

While this paper was under review, a new result with unquenched Wilson fermions appeared [18]. It uses the (unimproved) Wilson action and does not check the dependence on lattice spacing. It supports our conclusion that the real-world quark masses likely lie below the quenched approximation, but it presents two values for m_s, which differ from one another by nearly a factor of 2 (3 times their quoted uncertainty).

Finally, we place our results in the overall picture of existing determinations [2,3,6,17]. We disregard results with physical volumes smaller than 1.5 fm and lattice spacings larger than 0.2 fm (or 1.0 GeV^{-1}). We also disregard unquenched work with Wilson fermions, which is more primitive than that with staggered fermions. Of the remaining determinations, Fig. 2 shows that the cut-off dependence and resulting large size of quark masses from quenched Wilson fermions arise mostly from the well-known $O(a)$ error. The discrepancy between the quenched clover-improved fermion results and the quenched staggered fermion results is plausibly attributed to the apparent poor convergence of staggered fermion perturbation theory, and to the remaining cut-off dependence in the improved results. The difference between quenched and unquenched staggered fermion results is roughly what is expected. In summary, after making reasonable cuts, existing determinations are compatible, with discrepancies explained by cut-off or perturbative effects. Hence, our results for the light quark masses, including estimated corrections for the effects of light quark loops, are consistent with others based on lattice gauge theory.

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