Early Universe Test of Nonextensive Statistics

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Within an early Universe scenario, nonextensive thermostatistics is investigated on the basis of data concerning primordial helium abundance. We obtain first order corrections to the energy densities and weak interaction rates, and use them to compute the deviation in the primordial helium abundance. After comparing with observational results, we get $|q - 1| < 2.08 \times 10^{-5}$ as a bound for the nonextensive parameter. [S0031-9007(97)03975-6]

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For a variety of physical reasons much work is nowadays devoted to nonlinear formalisms. Among them we can single out nonextensive thermostatistics (NET) [1] as an active area of research. NET is based upon the following two postulates [2,3].

Postulate 1.—The entropy of a system that can be found with probability p_i in any of W different microstates i is given by

$$S_q = (q - 1)^{-1} \sum_{i=1}^{W} [p_i - p_i^q], \tag{1}$$

with q a real parameter. We have a different statistics for every possible q value. Of course,

$$\sum_i p_i = 1, \tag{2}$$

and it is easy to see that, for $q = 1$, one regains the Boltzmann-Gibbs form [2]. The resulting physics is extensive just for $q = 1$. Otherwise we are led into the realm of nonextensivity [1–4].

Postulate 2.—An experimental measurement of an observable A, whose expectation value in microstate i is a_i, yields the q-expectation value (generalized expectation value)

$$\langle A \rangle_q = \sum_{i=1}^{W} p_i^q a_i \tag{3}$$

for the observable A.

Unappealing as the above postulates may perhaps be considered, but it should strongly be stressed that these two statements have the rank of *axioms*. As such, their validity is to be decided *exclusively* not by vague discomfort feelings but by the comparison with experiment of the conclusions to which they lead. One such a test was recently reported in Refs. [5,6], where bounds to $|q - 1|$ were stablished using the cosmic blackbody radiation. Also, present day determination of Stephan-Boltzmann constant σ puts a similar constraint, for which the order of magnitude is $|q - 1| < 10^{-4}$. However, it was later noted [7] that in both cases, the application of thermodynamics to these contexts is strictly local, and thus, a nonviolation of nonextensivity in a large scale could not be sustained on this basis. In this communication, we intend to find bounds on nonextensivity not affected by such criticism, which we deem fair.

The phenomenal success of thermodynamics and statistical physics crucially depends upon certain mathematical relationships involving energy and entropy, and much work has been devoted to (i) show that many of these relationships are valid for *arbitrary* q and (ii) to find appropriate generalizations for the rest. In this vein we just mention that, by suitably maximizing (1), Curado and Tsallis [3] found that the whole mathematical (Legendre-transform based) structure of thermodynamics becomes *invariant* under a change of the q value (from unity to any other real number), while the connection of NET both with quantum mechanics and with information theory was established in [4], where it was shown that all of the conventional Jaynes-Boltzmann-Gibbs [8] results generalize to the Tsallis' environment. For more details see [1]. Of course, to verify that NET is useful, it is necessary to show that it appropriately describes certain physical systems with q values that are different from unity. Much work in this respect has been been performed recently. We may cite applications to astrophysical problems [9,10], to Lévi flights [11], to turbulence phenomena [12], to simulated annealing [13], etc. The interested reader is referred to [1] for additional references. Now, NET establishes a different (from the orthodox) fashion of doing "statistics," i.e., a nonconventional way of *counting*, that has proved to be useful in a variety of contexts. The difference is governed by the value of the Tsallis parameter q. It is clearly recommended to use data concerning diverse natural phenomena to estimate the q value with reference to different scenarios. In the present communication we shall attempt to use the early Universe helium abundance data so as to find one of such estimates.

For the canonical ensemble, (1) gives [4]

$$\hat{\rho} = \frac{1}{Z_q} [\hat{1} - (1 - q)\beta \hat{H}]^{\frac{1}{1-q}}, \tag{4}$$

as the appropriate density operator and

$$Z_q = \mathrm{Tr}[\hat{1} - (1 - q)\beta \hat{H}]^{\frac{1}{1-q}}, \tag{5}$$

as the associated generalized partition function [4]. Here, as usual, $\beta = 1/kT$ and $\hat{H}$ is the Hamiltonian of the system.

We shall focus our attention upon the $\beta(q-1) \to 0$ limit, in which a first order expansion allows for analytical computations. The expression of the generalized mean value of an operator was computed in this limit by Tsallis, Sa Barreto, and Loh [6]. When applied to particle number operators, the concomitant result reads

$$\langle \hat{n} \rangle_q = \langle \hat{n} \rangle_{BG} Z_{BG}^{q-1} \Bigg\{ 1 + (1-q)x \times \left[\frac{\langle \hat{n}^2 \rangle_{BG}}{\langle \hat{n} \rangle_{BG}} + \frac{x}{2} \times \left(\langle \hat{n}^2 \rangle_{BG} - \frac{\langle \hat{n}^3 \rangle_{BG}}{\langle \hat{n} \rangle_{BG}} \right) \right] \Bigg\}, \quad (6)$$

where x stands for ϵ/kT (ϵ is the energy of a single particle) and the symbol BG means *to be computed within Boltzmann-Gibbs' statistical tenets.*

With the standard values of $\langle \hat{n}^2 \rangle_{BG}$ and $\langle \hat{n}^3 \rangle_{BG}$, for fermions and bosons, the corrections to the energy density in the early Universe may be computed [14]. When the particles are highly relativistic, $T \gg m$, and nondegenerate $T \gg \mu$, we get

$$\rho_{\text{bosons}} = \frac{g_b}{2\pi^2} \int_0^\infty dE\, E^3 \langle \hat{n} \rangle_{\text{bosons},q}, \quad (7)$$

$$\rho_{\text{fermions}} = \frac{g_f}{2\pi^2} \int_0^\infty dE\, E^3 \langle \hat{n} \rangle_{\text{fermions},q}, \quad (8)$$

where $g_{b,f}$ stands for the degeneracy factor of each one of the species involved. Using (6), we finally obtain

$$\rho_{\text{total}} = \rho_{\text{bosons}} + \rho_{\text{fermions}} = \frac{\pi^2}{30} g T^4 + \frac{1}{2\pi^2} \times (40.02 g_b + 34.70 g_f) T^4 (q-1), \quad (9)$$

where $g = g_b + 7/8 g_f$. At high enough temperatures, the energy density of the Universe is essentially dominated by e^-, e^+, ν and ν. Interactions among these particles keep all of them at nearly the same temperature. Accordingly, we set $g_b = 2$ and $g_f = 2 + 2 + 2 \times 3$ and reach thus the final form of the Tsallis correction to the energy density, namely,

$$\rho_{\text{total}} = \rho_{\text{standard}} + 21.63 T^4 (q-1). \quad (10)$$

We turn now our attention to the details of the computation of those corrections due to the weak interaction rate. This rate allows one to compute the neutron abundance as the Universe evolves. We shall denote by $\lambda_{pn}(T)$ the rate for the weak processes to convert protons into neutrons and by $\lambda_{np}(T)$ the rate for the associated, reverse ones. Following the standard computations [15,16], it is possible to see that, for high enough temperatures, the weak interaction rate is $\Lambda(T) \simeq \lambda_{np} + \lambda_{pn} \simeq G_F^2 T^5$, with λ_{np} being related with λ_{pn} by the principle of detailed balance [16]: $\lambda_{np} = \exp(-Q/T)\lambda_{pn}$, $Q = m_n - m_p = 1.29$ MeV, and G_F the Fermi constant. We want to compute first order Tsallis corrections [$(q-1)$ order] to $\Lambda(T)$. To do this, we need to analyze the individual interaction rates, i.e., each one of the terms in the sum

$$\lambda_{np} = \lambda_{\nu+n\to p+e^+} + \lambda_{e^+ + n \to p + \hat{\nu}} + \lambda_{n \to p + e^- + \hat{\nu}} \quad (11)$$

given by [15],

$$\lambda_{\nu+n\to p+e^-} = A \int_0^\infty dp_\nu\, p_\nu^2 p_e E_e (1 - \langle \hat{n}_e \rangle) \langle \hat{n}_\nu \rangle, \quad (12)$$

$$\lambda_{e^+ + n \to p + \hat{\nu}} = A \int_0^\infty dp_e\, p_e^2 p_\nu E_\nu (1 - \langle \hat{n}_\nu \rangle) \langle \hat{n}_e \rangle, \quad (13)$$

$$\lambda_{n \to p + e^- + \hat{\nu}} = A \int_0^{p_0} dp_e\, p_e^2 p_\nu E_\nu (1 - \langle \hat{n}_\nu \rangle)(1 - \langle \hat{n}_e \rangle), \quad (14)$$

where A is a constant fixed by the experimental value of $\lambda_{n\to p+e^- + \hat{\nu}}$. In the preceding equations, we have to consider, of course, energy conservation as a boundary condition which relates E_ν and E_e. During the period of freezing out of weak interactions we are interested in, several approximations are in order [16].

(i) All temperatures involved in the present game will be taken as equal, $T_e = T_\gamma = T_\nu = T$, which ensures that reverse reactions have the same form as the direct ones.

(ii) We shall neglect Pauli factors $(1 - \langle \hat{n}_\nu \rangle)$ and $(1 - \langle \hat{n}_e \rangle)$ because the typical energies which contribute in the integrals for the rate are much bigger than the temperature. Even in the case of nonextensive statistics, Pauli blockings corrections are $1 - n_q = 1 - n_{BG}[1 - x^2 e^{-x}(q-1)/2]$, which are neglectable for $x \gg 1$ and a first order deviation from $q = 1$.

(iii) We neglect also the electron mass in (12) and (13).

With these approximations (12) and (13) become identical. Using $p_e = E_e = Q + E_\nu$ in (12), the standard result follows.

Passing now to the nonextensive context, we must consistently use the $\langle \hat{n} \rangle_q$ distribution functions. Performing the previous integrations, we obtain the leading order corrections terms in the fashion

$$\lambda_{\nu+n\to p+e^-} = \lambda^{\text{standard}}_{\nu+n\to p+e^-} + (480 T^5 + 2 \times 84 T^4 Q + 18 T^3 Q^2) \times (1-q) A, \quad (15)$$

where

$$\lambda^{\text{standard}}_{\nu+n\to p+e^-} = (4!\, T^2 + 2 \times 3!\, TQ + 2!\, Q^2) A T^3. \quad (16)$$

In order to get some fresh insight into the problem, we shall consider here only the first correction, proportional to T^5. A more detailed analysis of these weak rates and the neutron-proton abundance ratio they yield is considered elsewhere [17]. As explained in [16], the high temperature regime makes $\lambda_{np} \simeq 2 \times \lambda_{\nu+n\to p+e^-}$ and $\Lambda \simeq 2\lambda_{np}$. As a consequence, the change in the weak reaction rate adopts the form

$$\frac{\delta\Lambda}{A} = 1920 T^5 (1-q). \quad (17)$$

We have now all the ingredients of the nucleosynthesis recipe at our disposal. Basically, nucleosynthesis is the competition between the weak interaction rate and the expansion rate, given by the Hubble constant via the Einstein equations. The ^{4}He production may be estimated—in the standard big bang model—as

$$Y_p = \lambda \left(\frac{2x}{1+x} \right)_{t_f}, \tag{18}$$

where $\lambda = \exp[-(t_{\rm nuc} - t_f)/\tau]$ stands for the fraction of neutrons which decayed into protons between t_f and $t_{\rm nuc}$, with t_f ($t_{\rm nuc}$) the time of freeze out of the weak interactions (nucleosynthesis), τ the neutron mean lifetime, and $x = \exp(-Q/kT)$ the neutron to proton equilibrium ratio [18]. It is straightforward to compute the deviation produced in Y_p by a variation in T_f, and correspondingly, t_f. We get

$$\delta Y_p = Y_p \left[\left(1 - \frac{Y_p}{2\lambda} \right) \ln\left(\frac{2\lambda}{Y_p} - 1 \right) + \frac{-2t_f}{\tau_n} \right] \frac{\delta T_f}{T_f}, \tag{19}$$

where a radiation era relationship between time and temperature of the form ($T \propto t^{-\frac{1}{2}}$) is assumed and the one puts $\delta T_{\rm nuc} = 0$, because it is fixed by the binding energy of the deuteron. Similar studies concerning bounds on gravitational theories were analized in [19–21]. Considering now $Y_p = Y_p^{\rm obs} = 0.23$ and $\delta Y_p = 0.01$, which is the observational error [22], and standard values for the times and the mean life of neutron—which, in fact, is not modified at order $(q-1)$—, we must ask for

$$0.01 > 0.3766 \left| \frac{\delta T_f}{T_f} \right| \tag{20}$$

to be satisfied in order to get an estimate of primordial ^{4}He production compatible with observational data.

In order to obtain a value for $\frac{\delta T_f}{T_f}$ in nonextensive statistics, we equate

$$\Lambda \simeq \left(\frac{\dot a}{a} \right) = \sqrt{\frac{8\pi G}{3}\rho_{\rm total}}\,. \tag{21}$$

Adding up the corrections due to the changes in the energy density and in the weak interaction rate, our first order result up to $(q-1)$ reads

$$\left| \frac{\delta T_f}{T_f} \right| = 1276.4(q-1)\,, \tag{22}$$

which allows for a stringent bound, using (20), on the value of q

$$|q-1| < 2.08 \times 10^{-5}. \tag{23}$$

Even by enlarging the observational error in Y_p three times, we would get $|q-1| < 6.7 \times 10^{-5}$. In obtaining this bound we reach the main goal of the present communication: the early Universe physics places a bound upon the Tsallis parameter q. It is worth stressing that the measured value of Y_p comes from a sample which has been thoroughly mixed, at least during the life of the Galaxy. Thus, our estimate of a possible variation of q is based on a volume of the order of the horizon at the nucleosynthesis epoch and the present test avoids the locality problem pointed out in Ref. [7]. There, as previously stated, the authors claimed that it is impossible to obtain any conclusion about no violation of extensivity in a large scale on the basis of results from the cosmic blackbody spectrum. As our result pertains to those taking place during the early childhood of our Universe, nonextensivity is severely constrained upon all epochs of cosmic evolution, with separate, independently tested, observational evidence.

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