PHYSICAL REVIEW LETTERS

VOLUME 79 | 1 SEPTEMBER 1997 | NUMBER 9

Localization of One-Photon States

Corin Adlard, E. R. Pike, and Sarben Sarkar
Department of Physics, King's College London, Strand, London WC2R 2LS, United Kingdom
(Received 25 October 1996)

Single-photon states with arbitrarily fast asymptotic power-law falloff of energy density and photodetection rate are explicitly constructed. This goes beyond the recently discovered tenth-power law of the Hellwarth-Nouchi photon which itself superseded the long-standing seventh-power law of the Amrein photon. [S0031-9007(97)03953-7]

PACS numbers: 03.70.+k

Given any classical solution of the source-free Maxwell's equations it is possible to write down a corresponding quantum mechanical one-photon state

$$|\phi\rangle = \sum_{\lambda=\pm 1} \int \frac{d^3k}{(2\pi)^3} f_{(\lambda)}(\vec{k}) a^{\dagger}_{(\lambda)}(\vec{k})\,|0\rangle\,, \tag{1}$$

where $|0\rangle$ represents the vacuum, $f_{(\lambda)}(\vec{k})$ is a c-number function, and $a^{\dagger}_{(\lambda)}(\vec{k})$ is the creation operator for a single photon state with momentum $\vec{k}$ and helicity λ. This is a general wave-packet state. The transverse nature of the classical solution is reflected in the photon having only 2 (rather than 3) spin degrees of freedom (i.e., 2 helicities). In particular, we will show that

$$\sum_{\lambda} \vec{\varepsilon}_{(\lambda)}(\vec{k}) f_{(\lambda)}(\vec{k}) = \sqrt{\frac{\varepsilon_0 \omega_{\vec{k}}}{\hbar}}\,\tilde{\vec{A}}(\vec{k})\,, \tag{2}$$

where $\tilde{\vec{A}}(\vec{k})$ is the spatial Fourier transform of $\vec{A}(t,\vec{r})$, the classical solution for the vector potential, and $\vec{\varepsilon}_{(\lambda)}(\vec{k})$ is the polarization vector associated with helicity λ and wave vector $\vec{k}$. It has been known for a long time that the configuration space representation of $|\phi\rangle$ cannot have delta-function support or support in a finite region of space [1,2]. This prompted Jauch and Piron [3] and Amrein [4] to make a formal modification of the imprimitivity formulation of localizability of elementary particles to a generalized imprimitivity applicable to photons. It was believed (without proof) that such a construction gave the tightest isotropically localized single-photon state. Such a state can be shown to have an asymptotic $1/|\vec{r}|^7$ falloff of its energy density and photodetection rate [5]. The situation remained thus for about a decade until recently when a particular explicit solution of Maxwell's equation by Hellwarth and Nouchi [6] led to a one-photon state with an asymptotic falloff of $1/|\vec{r}|^{10}$ for the photodetection rate [7] at any finite time t. Consequently, it became clear that the state from the generalized imprimitivity construction was not the most isotropically localized possible. However, the question of the existence of a fundamental limit to the sharpness of the power-law falloff remained open. In this paper we will show that single-photon states with arbitrarily high powers of asymptotic falloff can be explicitly constructed.

The EDEPT solutions.—Following the same calculations as in the paper of Ziolkowski [8], we can obtain the (focused) EDEPT (electromagnetic directed-energy pulse-train) solutions for the classical vector potential, expressed in cylindrical coordinates, as the real and imaginary parts of

$$\vec{A}(\tau,\rho,z) = \frac{2\alpha\mu_0 g_0^{\alpha} \vec{e}_{\theta} \rho [g_1 + i(z-\tau)]^{\alpha-1}}{[r^2 - \tau^2 + i(g_2 - g_1)z - i(g_2 + g_1)\tau + g_1 g_2]^{\alpha+1}}\,, \tag{3}$$

where α is an integer, $r^2 = \rho^2 + z^2$, $\tau = ct$, and, in the notation of Ziolkowski, we have set $b = 0$, $\beta = g_0$, $z_0 = g_1$, and $a\beta = g_2$. As we see from (1) and (2) these solutions provide a convenient way of producing normalizable single-photon states with all the required transversality properties.

Note that the leading power in distance in the denominator of the above is $r^{2\alpha+2}$ and that in the numerator the leading power is only α at any finite time t. Thus there exist real or imaginary solutions for the vector potential which fall off asymptotically as $1/r^{\alpha+2}$ where α is odd for a real solution and even for an imaginary solution. The solution of Hellwarth and Nouchi is obtained from Eq. (1) with $\alpha = 1$. Since the electric and magnetic fields are the derivatives of the vector potential with respect to distance and time, these fields will also inevitably fall off asymptotically as $1/r^{\alpha+c}$. Here c is some small integer whose value depends on whether a real or imaginary solution is required. Indeed an explicit calculation shows that the electric and magnetic fields fall off asymptotically as a power which increases linearly with increasing α. These expressions are rather long, and so we do not give them here. Since we have not imposed an upper bound on α, other than it remain finite, we conclude that there is, in principle, no limit to the asymptotic rate of falloff of the electric or magnetic energy densities of these *classical* solutions of Maxwell's equations.

Arbitrarily localized one-photon states.—We shall now demonstrate that there exist one-photon states with the same electric and magnetic energy densities as the classical EDEPT solutions. First, the Hamiltonian density of a free electromagnetic field in vacuum can be written as

$$\mathcal{H}(t,\vec{r}) = \frac{\varepsilon_0}{2}\,\mathcal{N}[\vec{E}^2(t,\vec{r}) + c^2\vec{B}^2(t,\vec{r})], \tag{4}$$

where $1/\varepsilon_0\mu_0 = c^2$ and where $\mathcal{N}$ represents normal operator ordering and $\vec{E}$ and $\vec{B}$ are free operator fields given, in the Coulomb gauge, by the usual relations

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} \tag{5}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A}\,. \tag{6}$$

As usual, normal ordering is required to remove the divergent energy density of the vacuum. The vector potential (operator) may be written, in vacuum, as

$$\vec{A}(t,\vec{r}) = \sum_{\lambda=\pm1}\int \frac{d^3k}{(2\pi)^3}\, e^{-i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{r})} \times \sqrt{\frac{\hbar}{2\varepsilon_0\omega_{\vec{k}}}}\,\vec{\varepsilon}_{(\lambda)}(\vec{k})a_{(\lambda)}(\vec{k}) + \text{H.c.} \tag{7}$$

For the general one-photon state $|\phi\rangle$ the energy density is given by

$$\begin{aligned} u_{\text{qm}}(t,\vec{r}) &= \langle\phi|\mathcal{H}(t,\vec{r})|\phi\rangle \\ &= \frac{\hbar}{2}\sum_{\lambda\lambda'}\int \frac{d^3kd^3k'}{(2\pi)^6} f^*_{(\lambda)}(\vec{k})f_{(\lambda')}(\vec{k}')e^{i(\omega_{\vec{k}}-\omega_{\vec{k}'})t-i(\vec{k}-\vec{k}')\cdot\vec{r}}\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}} \\ &\quad\times[\vec{\varepsilon}^{\,*}_{(\lambda)}(\vec{k})\cdot\vec{\varepsilon}_{(\lambda')}(\vec{k}') + \hat{\vec{k}}\times\vec{\varepsilon}^{\,*}_{(\lambda)}(\vec{k})\cdot\hat{\vec{k}}'\times\vec{\varepsilon}_{(\lambda')}(\vec{k}')]. \end{aligned} \tag{8}$$

Furthermore, we require a single-photon state to be normalizable and so

$$\langle\phi|\phi\rangle = \sum_{\lambda}\int \frac{d^3k}{(2\pi)^3}\,|f_{(\lambda)}(\vec{k})|^2 < \infty\,. \tag{9}$$

Now, since Ziolkowski uses a classical theory, we would like to find a one-photon state, $f_{(\lambda)}(\vec{k})$, such that it has the same energy density as the Hellwarth-Nouchi solutions. So we must find a suitable momentum-space expression for a classical energy density. The classical version of (4) is the energy density

$$u_{\text{cl}}(t,\vec{r}) = \frac{\varepsilon_0}{2}[\vec{E}^2(t,\vec{r}) + c^2\vec{B}^2(t,\vec{r})], \tag{10}$$

where $\vec{E}$ and $\vec{B}$ are now classical fields, so do not require normal ordering. Now, defining the Fourier transforms of the (real) electric field by

$$\begin{aligned} \vec{E}(t,\vec{r}) &= \int \frac{d^3k}{(2\pi)^3}\,\tilde{\vec{E}}(\vec{k})e^{-i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{r})} \\ &= \int \frac{d^3k}{(2\pi)^3}\,\tilde{\vec{E}}^*(\vec{k})e^{i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{r})}, \end{aligned} \tag{11}$$

and similarly for the magnetic field, and using these expressions in (10), we find

$$u_{\text{cl}}(t,\vec{r}) = \frac{\varepsilon_0}{2}\int \frac{d^3kd^3k'}{(2\pi)^6}\, e^{i(\omega_{\vec{k}}-\omega_{\vec{k}'})t-i(\vec{k}-\vec{k}')\cdot\vec{r}} \times [\tilde{\vec{E}}^*(\vec{k})\cdot\tilde{\vec{E}}(\vec{k}') + c^2\tilde{\vec{B}}^*(\vec{k})\cdot\tilde{\vec{B}}(\vec{k}')]. \tag{12}$$

On substituting the definition of the Fourier transform of $\vec{A}(t,\vec{r})$

$$\vec{A}(t,\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\,\tilde{\vec{A}}(\vec{k})e^{-i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{r})} \tag{13}$$

into (5) and (6) we obtain

$$\tilde{\vec{E}}(\vec{k}) = i\omega_{\vec{k}}\tilde{\vec{A}}(\vec{k}) \tag{14}$$

and

$$c\tilde{\vec{B}}(\vec{k}) = i\omega_{\vec{k}}\hat{\vec{k}}\times\tilde{\vec{A}}(\vec{k})\,, \tag{15}$$

where we have written $\vec{k} = |\vec{k}|\hat{\vec{k}}$ and $\omega_{\vec{k}} = c|\vec{k}|$. So, substituting (14) and (15) into (12) we find

$$u_{\rm cl}(t,\vec r) = \frac{\varepsilon_0}{2}\int \frac{d^3k d^3k'}{(2\pi)^6}\,\omega_{\vec k}\omega_{\vec k'}e^{i(\omega_{\vec k}-\omega_{\vec k'})t - i(\vec k-\vec k')\cdot\vec r} \times [\tilde{\vec A}^*(\vec k)\cdot\tilde{\vec A}(\vec k') + \hat{\vec k}\times\tilde{\vec A}^*(\vec k)\cdot\hat{\vec k}'\times\tilde{\vec A}(\vec k')]\,. \tag{16}$$

On using the Coulomb gauge condition in (13) we find in momentum space

$$\vec k\cdot\tilde{\vec A}(\vec k) = 0\,, \tag{17}$$

that is, $\tilde{\vec A}(\vec k)$ is always orthogonal to $\vec k$. Thus, we may expand $\tilde{\vec A}(\vec k)$ in the basis of two orthonormal polarization vectors $\vec\varepsilon_{(\pm 1)}(\vec k)$ such that

$$\vec\varepsilon_{(\lambda)}(\vec k)\cdot\vec\varepsilon_{(\lambda')}(\vec k) = \delta_{\lambda\lambda'} \tag{18}$$

and, of course,

$$\vec\varepsilon_{(\lambda)}(\vec k)\cdot\hat{\vec k} = 0\,. \tag{19}$$

So, we may write

$$\tilde{\vec A}(\vec k) = \sum_{\lambda=\pm 1}\tilde A_{(\lambda)}(\vec k)\vec\varepsilon_{(\lambda)}(\vec k)\,. \tag{20}$$

Substituting this into (16) we find

$$u_{\rm cl}(t,\vec r) = \frac{\varepsilon_0}{2}\sum_{\lambda\lambda'}\int \frac{d^3k d^3k'}{(2\pi)^6}\,\tilde A^*_{(\lambda)}(\vec k)\tilde A_{(\lambda')}(\vec k')\omega_{\vec k}\omega_{\vec k'}e^{i(\omega_{\vec k}-\omega_{\vec k'})t - i(\vec k-\vec k')\cdot\vec r} \times [\vec\varepsilon^*_{(\lambda)}(\vec k)\cdot\vec\varepsilon_{(\lambda')}(\vec k') + \hat{\vec k}\times\vec\varepsilon^*_{(\lambda)}(\vec k)\cdot\hat{\vec k}'\times\vec\varepsilon_{(\lambda')}(\vec k')]\,. \tag{21}$$

Thus, comparing (8) and (21) we see that the classical and quantum electric and magnetic energy densities will be equal if the single-photon state is given in momentum space by

$$f_{(\lambda)}(\vec k) = \sqrt{\frac{\varepsilon_0\omega_{\vec k}}{\hbar}}\,\tilde A_{(\lambda)}(\vec k)\,. \tag{22}$$

It can be shown for EDEPT solutions that such $f_{(\lambda)}(\vec k)$ are normalizable. Thus, we have shown that there indeed exist single-photon states which have an arbitrarily fast power-law falloff of their energy densities.

We shall now demonstrate that the detection rate for the localized one-photon states also fall off with an arbitrarily fast power law. The detection rate of a single-photon state $|\phi\rangle$ for an ideal, pointlike photon detector at $\vec r$ may be shown to be proportional to

$$S^{ij}\langle\phi|E_i^{(-)}(t,\vec r)E_j^{(+)}(t,\vec r)|\phi\rangle\,, \tag{23}$$

where S^{ij} is the detector sensitivity; see, for example, Glauber [9]. Let us note from the paper of Ziolkowski that the electric field of the doughnut solutions is polarized such that its classical electric field is in the $\vec e_\theta$ direction. It may similarly be shown that the "photodetection amplitude" $\langle 0|\vec E^{(+)}(t,\vec r)|\phi\rangle$ for the corresponding photon state $|\phi\rangle$ will be in this direction. Using this observation, we find that the detection rate is proportional to the electric energy density

$$\varepsilon_0\langle\phi|\vec E^{(-)}(t,\vec r)\cdot\vec E^{(+)}(t,\vec r)|\phi\rangle\,. \tag{24}$$

Now, since the electric energy densities of our single-photon states are equal to those of the corresponding EDEPT solutions, the single-photon detection rates are simply proportional to the classical electric energy densities, which fall off asymptotically with an arbitrarily fast power law. Hence, we have demonstrated that by choosing α the detection rates of these single-photon states can be made to have a power-law falloff which is arbitrarily rapid.

One of us (C. A.) was supported by a Junior Research Studentship from the School of Physical Sciences and Engineering of King's College London.

[1] E. R. Pike and S. Sarkar, *The Quantum Theory of Radiation* (Oxford University Press, London, 1995).
[2] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949); A. S. Wightman, Rev. Mod. Phys. **34**, 845 (1962).
[3] J. M. Jauch and C. Piron, Helv. Phys. Acta **40**, 559 (1967).
[4] W. O. Amrein, Helv. Phys. Acta **8**, 2684 (1969).
[5] E. R. Pike and S. Sarkar, Phys. Rev. A **35**, 926 (1987).
[6] R. W. Hellwarth and P. Nouchi, Phys. Rev. E **54**, 889–895 (1996).
[7] S. Sarkar and E. R. Pike, in Proceedings of the Conferenceon Quantum Coherence and Interference in Fundamental and Applied Physics, Crested Butte, 1994 (unpublished).
[8] R. W. Ziolkowski, Phys. Rev. A **39**, 2005 (1989).
[9] R. Glauber, in *Quantum Optics and Electronics*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannodji (Gordon and Breach, New York, 1964).