Conformal Invariance and Cosmic Background Radiation

Ignatios Antoniadis,[1] Pawel O. Mazur,[2] and Emil Mottola[3]
[1]Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau, France
[2]Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208
[3]Theoretical Division, Los Alamos National Laboratory, MS B285, Los Alamos, New Mexico 87545
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The spectrum and statistics of the cosmic microwave background radiation (CMBR) are investigated under the hypothesis that scale invariance of the primordial density fluctuations should be promoted to full conformal invariance, allowing for deviations from naive scaling. The spectral index of the two-point function of density fluctuations is determined by the trace anomaly to be greater than one, implying less power at large distance scales than a Harrison-Zel'dovich spectrum. Conformal invariance also implies non-Gaussian statistics of the CMBR and determines the large angular dependence of its three-point correlations. [S0031-9007(97)03472-8]

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With the discovery of the anisotropy in the cosmic microwave background radiation (CMBR) [1], cosmology has accelerated its transition from a field based largely on speculation to one in which observational data can be brought to bear on our understanding of the Universe. The CMBR anisotropy is the most sensitive available probe of the primordial density fluctuations from which the large scale structure of the Universe arose. Since the pioneering work of Harrison and Zel'dovich [2] it has been reasonable to suppose that these primordial fluctuations were generated with a scale invariant spectrum during an early epoch in the history of the Universe at the threshold of its classical evolution. Inflationary models are a particular realization of this idea which provide a quantum origin to the fluctuations with a spectral index n very close to one [3]. However scale invariance itself is more general than this. As a newly emerging physical science, the time now seems ripe to examine the broader context and implications of scale invariant behavior for cosmology, as revealed in its more developed sister sciences.

Scale invariance was first introduced into physics in early attempts to understand the apparently universal behavior observed in turbulence and second order phase transitions, which are independent of the particular dynamical details of the system. The gradual refinement and development of this simple idea of universality led to the modern theory of critical phenomena, one of whose hallmarks is well-defined logarithmic deviations from naive scaling relations [4]. A second general feature of the theory is the specification of higher point correlation functions of fluctuations according to the requirements of conformal invariance at the critical point [5].

In the language of critical phenomena, the observation of Harrison and Zel'dovich [2] that the primordial density fluctuations should be characterized by a spectral index $n = 1$ is equivalent to the statement that the observable giving rise to these fluctuations has engineering or naive scaling dimension $\Delta_0 = 2$. Indeed, because the density fluctuations are related to metric fluctuations by Einstein's equations, this naive scaling dimension simply reflects the fact that the relevant coordinate invariant measure of metric fluctuations is the scalar curvature $\delta R \sim G\delta\rho$, which is second order in derivatives of the metric. Hence, the fluctuations in the density perturbations are tied to the scalar curvature and the two-point spatial correlations of both should behave like $|x - y|^{-4}$, or $|k|^1$ in Fourier space, according to simple dimensional analysis.

One of the principal lessons of the modern understanding of critical phenomena is that naive dimensional analysis does *not* fix the transformation properties of observables under conformal transformations at the fixed point. On the contrary, one should expect to find well-defined logarithmic deviations from naive scaling, corresponding to a (generally noninteger) dimension $\Delta \neq \Delta_0$. The deviation from naive scaling $\Delta - \Delta_0$ is the "anomalous" dimension of the observable due to critical fluctuations, which may be quantum or statistical in origin. Once Δ is fixed for a given observable the requirement of conformal invariance determines the form of its two- and three-point correlation functions up to an arbitrary amplitude, without reliance on any particular dynamical model.

Two-point correlations.—Consider first the two-point function of two observables $\mathcal{O}_\Delta$ with dimension Δ. Conformal invariance requires [4,5]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle \sim |x_1 - x_2|^{-2\Delta} \qquad (1)$$

at equal times in three dimensional flat spatial coordinates. In Fourier space this becomes

$$G_2(k) \equiv \langle \tilde{\mathcal{O}}_\Delta(k)\tilde{\mathcal{O}}_\Delta(-k)\rangle \sim |k|^{2\Delta-3}. \qquad (2)$$

Thus, we define the spectral index of this observable by

$$n \equiv 2\Delta - 3\,. \qquad (3)$$

In the case that the observable is the primordial density fluctuation $\delta\rho$, and in the classical limit where its anomalous dimension vanishes, $\Delta \to \Delta_0 = 2$, we recover the Harrison-Zel'dovich spectral index of $n = 1$.

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In order to convert the power spectrum of primordial density fluctuations to the spectrum of fluctuations in the CMBR at large angular separations we follow the standard treatment [6] relating the temperature deviation to the Newtonian gravitational potential φ at the last scattering surface, $\frac{\delta T}{T} \sim \delta\varphi$, which is related to the density perturbation in turn by

$$\nabla^2 \delta\varphi = 4\pi G \delta\rho \,. \tag{4}$$

Hence, in Fourier space,

$$\frac{\delta T}{T} \sim \delta\varphi \sim \frac{1}{k^2}\frac{\delta\rho}{\rho}\,, \tag{5}$$

and the two-point function of CMBR temperature fluctuations is determined by the conformal dimension Δ to be

$$\begin{aligned} C_2(\theta) &\equiv \left\langle \frac{\delta T}{T}(\hat{r}_1)\,\frac{\delta T}{T}(\hat{r}_2)\right\rangle \\ &\sim \int d^3k \left(\frac{1}{k^2}\right)^2 G_2(k) e^{ik\cdot r_{12}} \\ &\sim \Gamma(2-\Delta)\,(r_{12}^2)^{2-\Delta}, \end{aligned} \tag{6}$$

where $r_{12} \equiv (\hat{r}_1 - \hat{r}_2)r$ is the vector difference between the two positions from which the CMBR photons originate. They are at equal distance r from the observer by the assumption that the photons were emitted at the last scattering surface at equal cosmic time. Since $r_{12}^2 = 2(1-\cos\theta)r^2$, we find then

$$C_2(\theta) \sim \Gamma(2-\Delta)\,(1-\cos\theta)^{2-\Delta} \tag{7}$$

for arbitrary scaling dimension Δ.

Expanding the function $C_2(\theta)$ in multipole moments,

$$C_2(\theta) = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)c_\ell^{(2)}(\Delta)P_\ell(\cos\theta)\,, \tag{8}$$

$$c_\ell^{(2)}(\Delta) \sim \Gamma(2-\Delta)\sin[\pi(2-\Delta)]\,\frac{\Gamma(\ell-2+\Delta)}{\Gamma(\ell+4-\Delta)}\,, \tag{9}$$

shows that the pole singularity at $\Delta = 2$ appears only in the $\ell = 0$ monopole moment. This singularity is just the reflection of the fact that the Laplacian in (4) cannot be inverted on constant functions, which should be excluded. Since the CMBR anisotropy is defined by removing the isotropic monopole moment (as well as the dipole moment), the $\ell = 0$ term does not appear in the sum, and the higher moments of the anisotropic two-point correlation function are well defined for Δ near 2. Normalizing to the quadrupole moment $c_2^{(2)}(\Delta)$, we find

$$c_\ell^{(2)}(\Delta) = c_2^{(2)}(\Delta)\,\frac{\Gamma(6-\Delta)}{\Gamma(\Delta)}\,\frac{\Gamma(\ell-2+\Delta)}{\Gamma(\ell+4-\Delta)}\,, \tag{10}$$

which is a standard result [6,7]. Indeed, if Δ is replaced by $\Delta_0 = 2$ we obtain $\ell(\ell+1)c_\ell^{(2)}(\Delta_0) = 6c_2^{(2)}(\Delta_0)$, which is the well-known predicted behavior of the lower moments ($\ell \leq 30$) of the CMBR anisotropy where the Sachs-Wolfe effect should dominate.

Our conformal symmetry considerations up to this point are quite general and leave Δ undetermined. Let us discuss now a physical source of quantum fluctuations on cosmological distance scales that can lead to a deviation from the classical scale dimension $\Delta_0 = 2$. As has been known for some time, the quantum zero-point energy of massless fields is modified in curved space and gives rise to a nonvanishing trace of the energy-momentum tensor, called the trace anomaly [8]. This nonzero T_μ^μ couples to the spin-0 or conformal part of the spacetime metric and causes it to fluctuate as well. Physically this means that the local standard of distance in the line element ds^2 varies from point to point in space. By the equivalence principle there is no strictly local coordinate invariant observable that is sensitive to these conformal fluctuations. However the correlations between the fluctuations at *different* spacetime points grow logarithmically, and on the characteristic scale of the horizon their magnitude becomes comparable with the classical gravitational potential in (4). At still larger scales these fluctuations dominate and lead to a renormalization group fixed point of gravity which is infrared stable [9]. Conformal invariance is thereby restored by these very large scale gravitational fluctuations, but the correlations and statistics of the CMBR entering our horizon should bear their imprint in the form of logarithmic deviations of the scaling relations from their classical counterparts, i.e., $\Delta \neq \Delta_0$.

This line of reasoning determines the scaling dimension of an observable with classical dimension Δ_0 to be [10]

$$\Delta = 4\,\frac{\sqrt{1-2(4-\Delta_0)/Q^2} - \sqrt{1-8/Q^2}}{1-\sqrt{1-8/Q^2}}\,. \tag{11}$$

where Q^2 is the relevant coefficient of the original trace anomaly (the Gauss-Bonnet term). Hence consideration of the trace anomaly generated by the zero-point fluctuations of massless fields leads necessarily to well-defined quantum corrections to the naive scaling dimensions of observables in cosmology. In the limit $Q^2 \to \infty$, the effects of fluctuations in the metric are suppressed and one recovers the classical scaling dimension Δ_0,

$$\Delta = \Delta_0 + \frac{1}{2Q^2}\Delta_0(4-\Delta_0) + \dots\,. \tag{12}$$

We estimate Q^2 in light of what is presently known about the trace anomaly of massless fields in Eq. (13) below, but for practical purposes Q^{-2} may be regarded as simply a parameter characterizing the universality class of the conformal metric fluctuations, which has no reason to vanish and should be determined from the observations. From this slightly more general perspective, the conformal invariance considerations that lead to (11) are quite independent of any particular model of their origin.

In the analysis of physical observables in the conformal sector of gravity, the operator with the lowest nontrivial scaling dimension corresponds, in the classical limit, to the scalar curvature with $\Delta_0 = 2$ [10]. Since the fluctuations which dominate at large distances correspond to observables with lowest scaling dimensions, the conformal factor theory in this limit selects precisely Harrison's original choice.

With $\Delta_0 = 2$, we find a definite prediction for deviations from a strict Harrison-Zel'dovich spectrum according to Eqs. (3) and (11) in terms of the parameter Q^2. The resulting spectral index n is plotted as a function of Q^2 in Fig. 1. It is always greater than 1 (if $8 \leq Q^2 < \infty$), and for large Q^2 it behaves as $n = 1 + \frac{4}{Q^2} + \ldots$. Comparing to the results of the four year cosmic background explorer differential microwave radiometer (COBE DMR) data analysis of the power spectrum, $0.9 \lesssim n_{\rm obs} \lesssim 1.5$ [11], we find that $Q^2_{\rm obs} \gtrsim 12.4$ from Fig. 1.

From the theoretical side, the value of Q^2 for free conformally invariant fields is known to be [8,12]

$$Q^2 = \frac{1}{180}(N_S + \tfrac{11}{2}N_F + 62N_V - 28) + Q^2_{\rm grav}, \quad (13)$$

where N_S, N_F, and N_V are, respectively, the number of free scalars, Weyl fermions, and vector fields and $Q^2_{\rm grav}$ is the contribution of spin-2 gravitons, which has not yet been determined unambiguously. The -28 contribution is that of the conformal or spin-0 part of the metric itself. The main theoretical uncertainty in determining $Q^2_{\rm grav}$ is that the Einstein theory is neither conformally invariant nor free, so that a method for evaluating the strong infrared effects of spin-2 gravitons is required which is insensitive to ultraviolet physics. Such an analysis may be possible by numerical methods on the lattice, which would also provide a nontrivial consistency check of the existence of the infrared fixed point of quantum gravity with the predicted scaling relations [10]. A purely one-loop computation gives $Q^2_{\rm grav} \simeq 7.9$ for the graviton contribution [12]. Taking this estimate at face value and including all known fields of the standard model (SM) of particle physics (for which $N_F = 45$ and $N_V = 12$) we find

$$Q^2_{\rm SM} \simeq 13.2 \quad \text{and} \quad n \simeq 1.45, \quad (14)$$

which is intriguingly close to the observational bound.

A deviation of this sort from the Harrison-Zel'dovich spectrum has implications for galaxy formation. Indeed, a determination of Q^2 close to its present observational bound together with the COBE quadrupole normalization of the spectrum at large scales implies more power at shorter subhorizon scales where galaxies formed. For the value of the spectral index $n \simeq 1.45$, the power spectrum has an enhancement factor of $(H_0 \times 20\ \text{Mpc}/2h)^{-0.45} \simeq 4.6$ at the $20h^{-1}$ Mpc distance scale, relative to the $n = 1$ spectrum. This would lead to earlier formation of structure at the galactic and galactic cluster scales than in the case of a primordial $n = 1$ spectrum. However, the form and normalization of the evolved cluster mass function at these scales is very much model dependent and would need to be reanalyzed *ab initio* in each model to decide if increased power in the primordial spectrum of adiabatic density fluctuations can be reconciled with the observations of the matter anisotropy on this scale [6]. It is noteworthy that the conformal fixed point for gravity predicts a "blue" spectral index $n > 1$ (for $Q^2 > 8$), while most suggestions for modifying the Harrison-Zel'dovich spectrum, such as extended or power law inflation (which do so by reducing the effective inflation rate) lead to $n \leq 1$ [13].

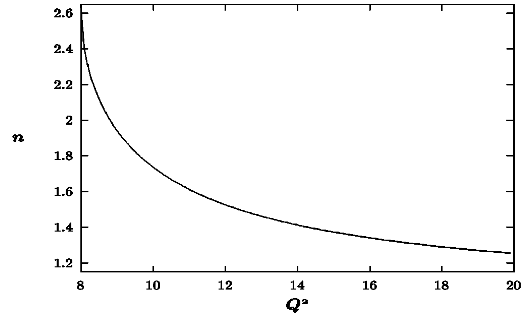

FIG. 1. The spectral index n as a function of Q^2.

Higher point correlations.—Turning now from the two-point function of CMBR fluctuations to higher point correlators, we find a second characteristic and unambiguous prediction of conformal invariance, namely non-Gaussian statistics for the CMBR. The first correlator sensitive to this departure from Gaussian statistics is the three-point function of the observable $\mathcal{O}_\Delta$, which takes the form [5]

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle \sim |x_1 - x_2|^{-\Delta}|x_2 - x_3|^{-\Delta}|x_3 - x_1|^{-\Delta}, \quad (15)$$

or in Fourier space,

$$G_3(k_1, k_2) \sim \int d^3p\,|p|^{\Delta-3}|p + k_1|^{\Delta-3}|p - k_2|^{\Delta-3} \sim \Gamma(3(1 - \tfrac{\Delta}{2}))\int_0^1 du \int_0^1 dv\, \frac{[u(1-u)v]^{1-\frac{\Delta}{2}}(1-v)^{-1+\frac{\Delta}{2}}}{[u(1-u)k_1^2 + v(1-u)k_2^2 + uv(k_1 + k_2)^2]^{3(1-\frac{\Delta}{2})}}. \quad (16)$$

This three-point function of primordial density fluctuations gives rise to three-point correlations in the CMBR by reasoning precisely analogous as that leading from Eqs. (2) to (6). That is,

$$C_3(\theta_{12}, \theta_{23}, \theta_{31}) \equiv \left\langle \frac{\delta T}{T}(\hat r_1)\frac{\delta T}{T}(\hat r_2)\frac{\delta T}{T}(\hat r_3)\right\rangle \sim \int \frac{d^3k_1 d^3k_2}{k_1^2k_2^2(k_1 + k_2)^2} G_3(k_1, k_2) e^{ik_1\cdot r_{13}} e^{ik_2\cdot r_{23}}, \quad (17)$$

where $r_{ij} \equiv (\hat r_i - \hat r_j)r$ and $r_{ij}^2 = 2(1 - \cos\theta_{ij})r^2$.

From the above expressions, it is easy to extract the global scaling of the three-point function in the infrared:

$$G_3(\lambda k_1, \lambda k_2) \sim \lambda^{3(\Delta-2)} G_3(k_1, k_2)\,, \qquad C_3 \sim r^{3(2-\Delta)}. \tag{18}$$

In the general case of three different angles, the expression for the three-point correlation function (17) is quite complicated, though it can be rewritten in parametric form analogous to (16) to facilitate numerical evaluation, if desired. An estimate of its angular dependence in the limit $\Delta \to 2$ can be obtained by replacing the slowly varying $G_3(k_1, k_2)$ by a constant. Then (17) can be evaluated by expanding in terms of spherical harmonics:

$$C_3(\theta_{ij}) \sim \sum_{l_i, m_i} \frac{\mathcal{K}^*_{l_1 m_1 l_2 m_2 l_3 m_3}}{(2l_1+1)(2l_2+1)(2l_3+1)} \times \left(\frac{1}{l_1+l_2+l_3} + \frac{1}{l_1+l_2+l_3+3}\right) \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3)\,, \tag{19}$$

where $\mathcal{K}_{l_1 m_1 l_2 m_2 l_3 m_3} \equiv \int d\Omega\, Y_{l_1 m_1}(\Omega) Y_{l_2 m_2}(\Omega) Y_{l_3 m_3}(\Omega)$.

In the special case of equal angles $\theta_{ij} = \theta$ [14], it follows from (18) that the three-point correlator is

$$C_3(\theta) \sim (1 - \cos\theta)^{\frac{3}{2}(2-\Delta)}. \tag{20}$$

Expanding the function $C_3(\theta)$ in multiple moments as in Eq. (8) with coefficients $c_\ell^{(3)}$, and normalizing to the quadrupole moment, we find

$$c_\ell^{(3)}(\Delta) = c_2^{(3)}(\Delta) \frac{\Gamma(4 + \frac{3}{2}(2-\Delta))}{\Gamma(2 - \frac{3}{2}(2-\Delta))} \times \frac{\Gamma(\ell - \frac{3}{2}(2-\Delta))}{\Gamma(\ell + 2 + \frac{3}{2}(2-\Delta))}. \tag{21}$$

In the limit $\Delta = 2$, we obtain $\ell(\ell+1)c_\ell^{(3)} = 6c_2^{(3)}$, which is the same result as for the moments $c_\ell^{(2)}$ of the two-point correlator but with a different quadrupole amplitude.

The value of this quadrupole normalization $c_2^{(3)}(\Delta)$ cannot be determined by conformal symmetry considerations alone. A naive comparison with the two-point function which has a small amplitude of the order of 10^{-6} leads to a rough estimate of $c_2^{(3)} \sim \mathcal{O}(10^{-9})$, which would make it very difficult to detect [14]. However, if the conformal invariance hypothesis is correct, then these non-Gaussian correlations must exist at some level, in distinction to the simplest inflationary scenarios. Their amplitude is model dependent and possibly much larger than the above naive estimate. The detection of such non-Gaussian correlations (or non-Gaussian statistics of the two-point correlator) at any level is therefore an important test for the hypothesis of conformal invariance.

For higher point correlations, conformal invariance does not determine the total angular dependence. Already the four-point function takes the form

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle \sim \frac{A_4}{\prod_{i<j} r_{ij}^{2\Delta/3}}\,, \tag{22}$$

where the amplitude A_4 is an arbitrary function of the two cross ratios, $r_{13}^2 r_{24}^2 / r_{12}^2 r_{34}^2$ and $r_{14}^2 r_{23}^2 / r_{12}^2 r_{34}^2$. Analogous expressions hold for higher p-point functions. However, in the equilateral case $\theta_{ij} = \theta$, the coefficient amplitudes A_p become constants and the angular dependence is again completely determined. The result is

$$C_p(\theta) \sim (1 - \cos\theta)^{\frac{p}{2}(2-\Delta)}, \tag{23}$$

and the expansion in multiple moments yields coefficients $c_\ell^{(p)}$ of the same form as in Eq. (21) with $3/2$ replaced by $p/2$. In the limit $\Delta = 2$, we obtain the universal ℓ-dependence $\ell(\ell+1)c_\ell^{(p)} = 6c_2^{(p)}$.

In summary, the conformal invariance hypothesis applied to the primordial density fluctuations predicts deviations from the Harrison-Zel'dovich spectrum, which should be imprinted on the CMBR anisotropy. A particular realization of this hypothesis is provided by the metric fluctuations induced by the known trace anomaly of massless matter fields which gives rise to a fixed point with a spectral index $n > 1$. A second general consequence of conformal invariance is non-Gaussian statistics which fix in particular the form of the three-point correlations of the CMBR. If either of these effects is detected it would require a reappraisal of the current models of the origin of primordial density fluctuations and the formation of structure in the Universe.

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