Magnus and Iordanskii Forces in Superfluids

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The transverse force acting on a quantized vortex in a superfluid is a problem that has eluded a complete understanding for more than three decades. In this Letter I calculate the *superfluid* velocity part of the transverse force in a way closely related to Laughlin's argument for the quantization of conductance in the quantum Hall effect. A combination of this result, the *vortex* velocity part of the transverse force found by Thouless, Ao, and Niu [Phys. Rev. Lett. **76**, 3758 (1996)], and Galilean invariance shows that there cannot be a transverse force proportional to the normal fluid velocity. [S0031-9007(97)03821-0]

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The importance of quantized vortices in superfluids has been recognized since Onsager first put forth the idea of quantization of circulation almost 50 years ago [1,2]. A vortex moving in a superfluid experiences a force transverse to its velocity which is equivalent to the Magnus or Kutta-Joukowski hydrodynamic lift present in classical hydrodynamics [3], which is generally written

$$\mathbf{F} = \rho\, \boldsymbol{\kappa} \times (\mathbf{v}_{\rm v} - \mathbf{v}_{\rm fluid}), \qquad (1)$$

per unit length of the vortex [4]. Here ρ is the mass density of the fluid, $(\mathbf{v}_{\rm v} - \mathbf{v}_{\rm fluid})$ is the velocity of the vortex relative to the fluid, and $\boldsymbol{\kappa}$ is a vector in the direction of the vortex with magnitude equal to the circulation ($\kappa = \oint \mathbf{v}_{\rm fluid} \cdot d\mathbf{l}$).

However, there is no consensus on the problem of generalizing the Magnus force to the superfluid case, and various inequivalent expressions for the relevant forces can be found in the literature [5–11].

This Letter deals specifically with the calculation of the *superfluid velocity part of the Magnus force*, which is the transverse force that depends on the superfluid velocity $\mathbf{v}_{\rm s}$. It is shown that this force is given by

$$\mathbf{F}_{\rm s} = -\rho_{\rm s}\, \boldsymbol{\kappa}_{\rm s} \times \mathbf{v}_{\rm s}, \qquad (2)$$

where $\rho_{\rm s}$ is the density of the superfluid component [12], and $\kappa_{\rm s}$ is the circulation of the superfluid around a vortex (a multiple of the quantum of circulation h/m [1,2]).

Recently Thouless, Ao, and Niu [5] (TAN) have convincingly argued for a universal *vortex velocity part of the Magnus force*

$$\mathbf{F}_{\rm v} = \rho_{\rm s}\, \boldsymbol{\kappa}_{\rm s} \times \mathbf{v}_{\rm v} \qquad (3)$$

per unit length for uniform neutral superfluids; and Geller, Wexler, and Thouless have generalized TAN's results to charged systems in the presence of a periodic potential [13]. The natural assumption by TAN that the normal fluid does not circulate around the vortex ($\kappa_{\rm n} = \oint \mathbf{v}_{\rm n} \cdot d\mathbf{l} = 0$) has been confirmed by a recent calculation [11] in the thermodynamic limit (mean free path of excitations much smaller than system size). There is an important point to notice: TAN deals only with the part of the transverse force that depends on the *vortex* velocity, while no statement is made regarding the other parts of this force that depend on the normal and superfluid velocities.

The confusion in the topic of writing the various parts of the Magnus force is widespread. Part of it arises from different interpretations on the role played by excitations, and whether or not these are scattered asymmetrically by the vortex [14] leading to a transverse force (the Iordanskii force) proportional to the normal fluid density $\rho_{\rm n}$ and either the relative velocities $(\mathbf{v}_{\rm n} - \mathbf{v}_{\rm v})$ or $(\mathbf{v}_{\rm n} - \mathbf{v}_{\rm s})$, where $\mathbf{v}_{\rm n}$ is the velocity of the normal component far from the vortex.

The results presented in this Letter, combined with TAN and Galilean invariance, are incompatible with the existence of a transverse force proportional to the normal fluid velocity $\mathbf{v}_{\rm n}$.

I must note that this Letter does not deal with the determination of additional dissipative terms (namely the *longitudinal* forces), which are negligible under the conditions of the present work. This is an interesting subject as well, but the arguments presented determine only the transverse forces.

In Section I I write down the most general transverse force, linear in the velocities, which is compatible with Galilean invariance. Section II is the main section of the Letter, where the superfluid velocity part of the Magnus force is calculated. In Section III I write the final form of the transverse force by combining the results of this Letter and TAN, plus Galilean invariance. I also discuss the diverse results obtained by other authors and their assumptions.

I. Galilean invariant transverse force.—In a homogeneous superfluid the forces acting on a vortex must be expressed in terms of velocity differences only. Consider a rectilinear vortex moving with velocity $\mathbf{v}_{\rm v}$ in a superfluid where the superfluid component has an asymptotic velocity $\mathbf{v}_{\rm s}$ and the normal component $\mathbf{v}_{\rm n}$. The most general *Galilean invariant transverse force* can be written as

$$\mathbf{F} = A\, \hat{\boldsymbol{\kappa}} \times (\mathbf{v}_{\rm v} - \mathbf{v}_{\rm s}) + B\, \hat{\boldsymbol{\kappa}} \times (\mathbf{v}_{\rm v} - \mathbf{v}_{\rm n}), \qquad (4)$$

where A and B are constants to be determined and $\hat{\boldsymbol{\kappa}}$ is a unit vector pointing in the direction of the vortex line. Our task is the determination of these unknown coefficients. It is customary to divide this expression into separate terms, each involving one particular velocity and denote them accordingly: the *vortex velocity* part of the Magnus force $\mathbf{F}_\mathrm{v} = (A + B)\, \hat{\boldsymbol{\kappa}} \times \mathbf{v}_\mathrm{v}$, the *superfluid velocity* part $\mathbf{F}_\mathrm{s} = (-A)\, \hat{\boldsymbol{\kappa}} \times \mathbf{v}_\mathrm{s}$, and the *normal fluid velocity* part $\mathbf{F}_\mathrm{n} = (-B)\, \hat{\boldsymbol{\kappa}} \times \mathbf{v}_\mathrm{n}$. Knowledge of two of these forces completely determines the third. In the following section I determine the coefficient A by calculating the superfluid velocity part of the Magnus force.

II. Free energy and force.—Here I wish to present a very simple *gedanken* experiment, whose outcome will determine the coefficient A as mentioned above. This argument has some parallels to Laughlin's own thought experiment relating to the quantization of Hall conductance in the quantum Hall effect [15].

Consider a neutral superfluid trapped inside a toroid like the one shown in Fig. 1. For simplicity assume the toroid to have a roughly uniform section and that the circumference L_x is much bigger than L_y. This makes the superfluid velocity v_s approximately uniform, which is what is actually desired for a definition of the *superfluid velocity* part of the Magnus force. I must remark that the argument is more general, and these assumptions are merely necessary to keep the argument clean and simple.

Assume that in the initial state $N \gg 1$ quanta of circulation are trapped in the toroid so that the superfluid velocity is given by $v_\mathrm{s} = Nh/mL_x$. Under this condition the normal fluid is pinned to the container and v_n is zero (in fact, this is how the normal density ρ_n is normally *defined* [12,16]). At some initial time a vortex is created at the outer edge and *slowly* dragged toward the center of the ring by some means [5], where it is annihilated at a later time $t = \tau \to \infty$. The final state corresponds to a trapped circulation $(N + 1)\, h/m$, while the normal fluid will still be at rest. By performing this process very slowly, dissipative effects are negligible.

While transporting the vortex across the ring, one needs to perform work on the system. In terms of Eq. (4), and given the fact that $v_\mathrm{v} = v_\mathrm{n} = 0$, the work is given by integrating the force per unit length along the displacement $d\mathbf{r}$ of each vortex segment $d\mathbf{l}$:

$$\begin{aligned} W &= -\int A\,(d\mathbf{l} \times \mathbf{v}_\mathrm{s}) \cdot d\mathbf{r} = A \int (d\mathbf{l} \times d\mathbf{r}) \cdot \mathbf{v}_\mathrm{s} \\ &= A \int d\mathbf{S} \cdot \mathbf{v}_\mathrm{s} = A\,(L_y L_z)\, v_\mathrm{s}. \end{aligned} \tag{5}$$

For *isolated* systems the change in energy corresponds to the amount of work. The argument is straightforward, but it is much simpler to consider an *isothermal* process. The amount of work performed then corresponds to the variation of the Helmholtz *free* energy $\mathcal{A} = \mathcal{E} - TS$ [17]. The free energy can be expressed in terms of the energy of the ground state plus the free energy of excitations:

$$\mathcal{A} = E_\mathrm{g.s.} + \mathcal{A}_\mathrm{excit}\,. \tag{6}$$

One will be interested in the variation of the free energy due to a variation of the superfluid velocity v_s; therefore, one needs only to consider the v_s dependent portions of $\mathcal{A}$. The relevant ground state energy is

$$E_\mathrm{g.s.} = (L_x L_y L_z)\rho\; v_\mathrm{s}^2/2\,, \tag{7}$$

and the excitation free energy is given by the standard expression

$$\mathcal{A}_\mathrm{excit} = (k_B T) \sum_\mathrm{modes} \ln(1 - e^{-\epsilon/k_B T})\,, \tag{8}$$

where the excitation energies ϵ are those in the "rest frame", and are therefore Doppler shifted by the superflow:

$$\epsilon = \hbar\omega(k) + v_\mathrm{s} \hbar k_x\,. \tag{9}$$

To second order in the superfluid velocity the free energy is given by

$$\mathcal{A}_\mathrm{excit} - \mathcal{A}_\mathrm{excit}(v_\mathrm{s} = 0) = \frac{v_\mathrm{s}^2}{2} \left.\frac{\partial^2 \mathcal{A}_\mathrm{excit}}{\partial v_\mathrm{s}^2}\right|_{v_s = 0} = -\frac{\hbar^2 v_\mathrm{s}^2}{2k_B T} \sum_\mathrm{modes} k_x^2 \frac{e^{\hbar\omega(k)/k_B T}}{(e^{\hbar\omega(k)/k_B T} - 1)^2} = -(L_x L_y L_z)\,\rho_\mathrm{n}\,\frac{v_\mathrm{s}^2}{2}\,, \tag{10}$$

where the last equality follows from the usual Landau derivation of the normal density [12,16].

The total change in Helmholtz free energy for variations in the superfluid velocity can be written as [18]

$$\begin{aligned} \Delta\mathcal{A} &= (L_x L_y L_z) \frac{\rho - \rho_\mathrm{n}}{2}\; \Delta(v_\mathrm{s}^2) \\ &= (L_x L_y L_z) \frac{\rho_\mathrm{s}}{2} \Delta(v_\mathrm{s}^2) = (L_y L_z)\;\rho_\mathrm{s}\; v_\mathrm{s}\,\frac{h}{m}\,, \end{aligned} \tag{11}$$

where the last equality corresponds to the change $\Delta v_\mathrm{s} = h/mL_x$ due to the motion of the vortex across the ring.

By equating the work performed on the system [Eq. (5)] and the variation of free energy [Eq. (11)] the unknown coefficient A in the general expression for the transverse force (4) can be determined:

$$A = \rho_\mathrm{s} \frac{h}{m}\,. \tag{12}$$

III. Total transverse force and conclusions.—Having calculated the *superfluid* velocity part of the transverse force, there is the need to obtain one more component of it to completely determine the transverse force (4).

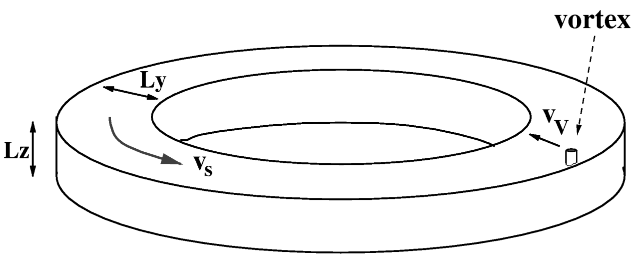

FIG. 1. *Gedanken* experiment: a vortex is created at the outer edge of a toroid with circumference L_x, adiabatically transported across the channel and annihilated at the inner edge, thus increasing the total circulation around the toroid by one quantum of circulation h/m.

By considering TAN's result [5,11], as written in Eq. (3), it is clear that the coefficient of the *vortex* velocity $(A + B)$ is given by

$$(A + B) = \rho_s \frac{h}{m}, \tag{13}$$

and this, along with the result for A calculated in the previous section, yields unambiguously $B = 0$, meaning that there is no transverse force depending on the *normal fluid* velocity. The total *transverse* force per unit length acting on a vortex can be written

$$\mathbf{F} = \rho_s \frac{h}{m} \hat{\mathbf{z}} \times (\mathbf{v}_V - \mathbf{v}_s), \tag{14}$$

with the transverse Iordanskii force vanishing exactly.

This is in general agreement with some direct calculations of the normal fluid velocity part of the transverse force based on the scattering of excitations by the vortex [9,19]. These calculations also show that the coefficient of the normal fluid velocity either vanishes or is much smaller than previously thought, in apparent conflict with Iordanskii's theory of the transverse force on a vortex [10,14].

It is interesting to note that while this Letter, in combination with TAN's result, yields an *exactly* vanishing Iordanskii force, the "direct" calculations mentioned above either give a nonzero result [10,14] or can only hint that it is small [9,11].

One must emphasize some differences in the assumptions about the asymptotic flow of excitations far away from the vortex. The authors calculating directly the Iordanskii force from the excitation scattering assume a homogeneous distribution of noninteracting phonons [9,10,14,19], while the calculations used along this Letter and references [5] and [11] include the effect of the vortex in the excitation distribution. This can explain some of the apparent discrepancies: a careful calculation of the Iordanskii force in the *hydrodynamic* regime (where the mean free path of excitations is much smaller than the size of the system) must include *both* the effects of the scattering of these excitations *and* the perturbation of the distribution functions by the vortex. Similar effects have been long known in the calculation of the viscous drag of a moving object in a fluid, namely the Stokes problem [20]. It may be possible to obtain independently the exact cancellation of the transverse Iordanskii force by including all the effects described in this paragraph. While this would be certainly desirable, it goes beyond the scope of this Letter.

At very low temperatures, however, the mean free path increases dramatically [12]. Excitations move ballistically, equilibrating primarily with the walls of the container and perturbations of the distribution functions by the moving vortex may be neglected. A direct calculation of the Iordanskii force based on the scattering of excitations by the vortex should be valid, yet there is no consensus on the magnitude of the force calculated in this manner; some [10,14] obtain a considerable force $(-\rho_n \boldsymbol{\kappa} \times \mathbf{v}_n)$, while others [9,19] find a result much smaller than this. I should emphasize that the argument presented in this Letter (Sect. II) for the superfluid velocity part of the transverse force is still valid in this regime, since the formal definition of the normal density and velocity in terms of the momentum density or free energy are unmodified [16] (although the actual expression for the normal density ρ_n will be different, in general anisotropic and dependent on the channel geometry). The absence of the normal fluid circulation around the vortex needed for TAN's result will also be valid but trickier, given the complicated geometry dependence. The importance of the distinction at temperatures corresponding to the ballistic regime is relative; the normal density ρ_n is extremely small and the superfluid density ρ_s is essentially indistinguishable from the total density ρ [12].

In conclusion, we have obtained the superfluid velocity part of the transverse force on a vortex using a robust and simple thermodynamic argument. A combination of this result, the vortex velocity part of the transverse force found by Thouless, Ao, and Niu [5], and Galilean invariance implies that there cannot be any transverse force dependent on the normal fluid velocity.

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