Generalization of Eshelby's Formula for a Single Ellipsoidal Elastic Inclusion to Poroelasticity and Thermoelasticity

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Eshelby's formula gives the response of a single ellipsoidal elastic inclusion in an elastic whole space to a uniform strain imposed at infinity. Using a linear combination of results from two simple thought experiments, we show how this formula may be generalized to both poroelasticity and thermoelasticity. The resulting new formulas are important for applications to analysis of poroelastic and thermoelastic composites, including but not restricted to rocks. [S0031-9007(97)03787-3]

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Probably the single most referenced work in the extensive and rapidly growing literature on elastic composites is Eshelby's paper [1] on the response of a single ellipsoidal elastic inclusion in an elastic whole space to a strain imposed at infinity. Eshelby found that a uniform strain at infinity results in a uniform strain within the ellipsoidal inclusion. This simple fact was then used in more detailed calculations to obtain the fourth-rank tensors relating these two uniform strains. The tensors themselves are not simple in general since they involve elliptic integrals, but Eshelby was able to enumerate and explicitly evaluate all of these integrals for simple shapes like spheres, oblate and prolate spheroids, needles, and disks. The results have been found to be immensely useful in the analysis of composite materials, since most inclusion shapes commonly of interest can be approximated by some ellipsoid. Effective medium theories [2–8] for elastic constants have very often been based on static or dynamic approximations that make explicit use of Eshelby's formulas. Recent reviews of effective medium theories and rigorous bounding methods applied to composite analysis are available [9–10].

It would be of considerable interest to have an identity analogous to Eshelby's result available in other, more complex, problems in composites analysis (e.g., piezoelectric composites [11]). Two problems that are themselves relatively straightforward generalizations of elasticity are poroelasticity and thermoelasticity. In poroelasticity, we allow the possibility that the elastic materials contain connected voids or pores and that these pores may be filled with fluids under pressure which then couples to the mechanical effects of an externally applied stress or strain. In thermoelasticity, we include the effects of temperature on the elastic materials and consider the coupling between thermal expansion and externally applied stresses and strains. In fact, it is known that problems in these two subjects have very similar mathematical structure [12–15], so that solutions found in one generally carry over with only minor modifications to the other. The main purpose of this paper is to show that the results of two simple thought experiments can be combined to produce a rigorous generalization of Eshelby's formula valid for either poroelasticity or thermoelasticity. Then, the hard part of Eshelby's work in computing the elliptic integrals (needed to evaluate the fourth-rank tensors) can be carried over to these new results with only trivial modifications.

We will first discuss the problem in terms of poroelasticity and later point out the modifications necessary to map onto thermoelasticity. In our notation, a superscript i refers to the inclusion phase, while superscripts h and $*$ refer to host and composite media, respectively. In this application the composite is a very simple one, being an infinite medium of host material with a single ellipsoidal inclusion of the ith phase. The basic result of Eshelby is then of the form

$$e_{pq}^{(i)} = T_{pqrs} e_{rs}^{*}, \tag{1}$$

where $e^{(i)}$ is the uniform induced strain in the inclusion, e^* is the uniform applied strain of the composite at infinity, and T is the fourth-rank tensor relating these two strains [4]. The summation convention for repeated indices is assumed in expressions such as (1). In elasticity, the components of T depend explicitly on the elastic constants of both the host and inclusion.

In our first thought experiment, we consider that in the absence of pore-fluid effects in poroelasticity (or thermal effects in thermoelasticity), the formula (1) must remain unchanged. In poroelasticity the only difference induced by the generalization from elasticity is an implicit one arising from the interpretation of the elastic constants used in evaluating the fourth-rank tensor T. Two types of bulk and shear moduli must be considered in poroelasticity, frame moduli (of the overall porous medium, often called the "frame") and grain or mineral moduli (of the purely solid constituents). In the absence of any pore fluid, only external confining stresses are operative and the only pertinent moduli are the frame moduli, corresponding to moduli one would measure for a porous sample of the material drained of all fluid. We will use the symbols K^*, $K^{(h)}$, and $K^{(i)}$ for the frame bulk moduli of the composite, the host, and the inclusion, respectively. The frame shear moduli are given similarly by μ^*, $\mu^{(h)}$, and $\mu^{(i)}$. The host and inclusion frame moduli [16] are the only ones that can appear in the expressions for the tensors T in poroelasticity.

The general relations between strains and stresses in an isotropic poroelastic medium take the form

$$e_{pq}^{(h)} = S_{pqrs}^{(h)}\sigma_{rs} + \frac{\alpha^{(h)}}{3K^{(h)}} p_f \delta_{pq}, \qquad (2)$$

where σ is the applied external stress, $S^{(h)}$ is the compliance tensor of the host frame material, and $\alpha^{(h)}$ is the Biot-Willis parameter [17] of the host medium. When a uniform saturating fluid is present in the pores of a microhomogeneous poroelastic medium, the resulting uniform strains in an isotropic medium are related to applied uniform (hydrostatic) stresses by

$$-e_{pq}^{(h)} = \left[\frac{p_c - p_f}{3K^{(h)}} + \frac{p_f}{3K_m^{(h)}}\right]\delta_{pq} = \frac{p_c - \alpha^{(h)}p_f}{3K^{(h)}}\,\delta_{pq}, \qquad (3)$$

where $p_c = -\frac{1}{3}\sigma_{ss}$ is a uniform external (at infinity in these inclusion problems) confining pressure (positive under compression) and $K_m^{(h)}$ is the grain or mineral bulk modulus of the host material. From (2) and (3), the Biot-Willis parameter [17] is seen to be given by $\alpha^{(h)} = 1 - K^{(h)}/K_m^{(h)}$. Expressions similar to (3) apply to e_{pq}^* and $e_{pq}^{(i)}$, with the corresponding changes in the bulk moduli and other parameters.

Now for our second thought experiment, we consider under what circumstances the host medium and the ellipsoidal inclusion will expand or contract at the same rate. See Fig. 1. This scenario is possible in poroelasticity because there are two adjustable fields present. (No such possibility exists in the purely elastic problem of Eshelby.) Analogous problems were first discussed originally in thermoelasticity [18–20] and more recently in poroelasticity [13]. The trick is that, if a ratio of p_c and p_f can be found so that $e^{(h)}$ and $e^{(i)}$ change at the same rate, then so must e^* and, furthermore, no local concentrations of stress develop. The resulting macroscopic strains are uniform; the macroscopic stresses are uniform; and, therefore, stress equilibrium conditions are trivially satisfied. Thus, the entire analysis of these stress states reduces to simple algebra. We know from earlier work [13] that the uniform expansion/contraction ratio can be found for any two-phase poroelastic composite, and the single ellipsoidal inclusion example considered here is just an especially simple two-phase problem.

In the uniform expansion/contraction scenario, once the pore pressure p_f (which is uniform throughout host and inclusion because of assumed open-pore boundary conditions) has been specified, then we know that the confining pressure p_c needed to produce a uniformly expanded or contracted state is given by

$$p_c/p_f = \frac{(\alpha^{(h)}/K^{(h)} - \alpha^{(i)}/K^{(i)})}{1/K^{(h)} - 1/K^{(i)}} \equiv R, \qquad (4)$$

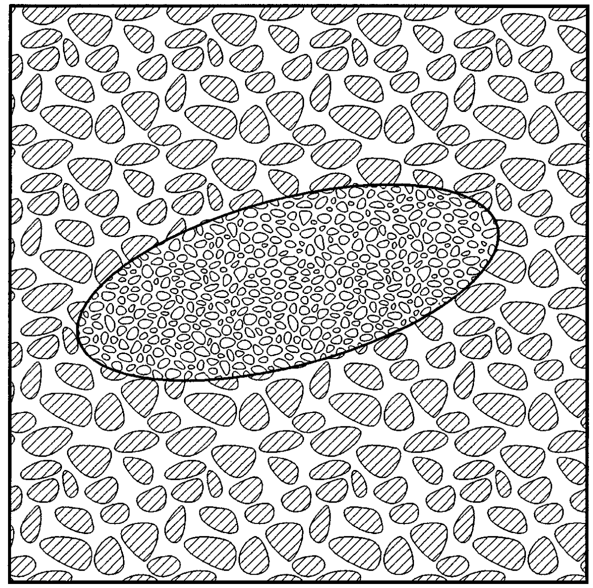

FIG. 1. An ellipsoidal inclusion of one poroelastic medium imbedded in another poroelastic medium. Assumed boundary conditions are welded contact between porous solid frames and open pores for the saturating fluid. The same figure applies (although with different assumed level of magnification) to both the initial unstrained state and to the state of uniform expansion/contraction in the second thought experiment.

depending only on the (assumed known) physical properties of the host and inclusion. This result was obtained by setting $e^{(h)} = e^{(i)}$ and solving for the p_c/p_f ratio.

Then, the strains of the reference states of the composite, host, and inclusion materials are given by

$$\varepsilon_{pq}^*(p_f) = -\frac{p_f}{3K^*}(R - \alpha^*)\delta_{pq}, \qquad (5)$$

$$\varepsilon_{pq}^{(h)}(p_f) = -\frac{p_f}{3K^{(h)}}(R - \alpha^{(h)})\delta_{pq}, \qquad (6)$$

and

$$\varepsilon_{pq}^{(i)}(p_f) = -\frac{p_f}{3K^{(i)}}(R - \alpha^{(i)})\delta_{pq}, \qquad (7)$$

all of which are equal $\varepsilon_{pq}^* = \varepsilon_{pq}^{(h)} = \varepsilon_{pq}^{(i)} = \delta_{pq}(p_f/3)(\alpha^{(h)} - \alpha^{(i)})/(K^{(h)} - K^{(i)})$, by construction. The ratio R is the one defined in (4), which is easily verified by equating (6) and (7) and solving for R. [The remaining equality among (5)–(7) determines the value of α^* of the composite [13].]

Thus, the final form of the generalization of Eshelby's formula to poroelasticity is given by

$$e_{pq}^{(i)} - \varepsilon_{pq}^{(i)} = T_{pqrs}(e_{rs}^* - \varepsilon_{rs}^*). \qquad (8)$$

We see that, if the pore-fluid pressure vanishes (e.g., $p_f = 0$ in the absence of a pore fluid), then the uniform strains ε disappear from (8), and it reduces exactly to (1) as it should. For the other limiting case, when the pore pressure has been specified to be $p_f \neq 0$, then the uniform strains ε in (8) can be computed from (5) and (7). Now, if the strain at infinity happens to be chosen to be equal to this uniform strain, (8) shows that the inclusion strain takes the value at infinity as it should. Since the equation for $e^{(i)}$ is linear, these two cases are enough to determine the behavior for arbitrary values of e^* and p_f.

The deceptively simple equation (8) is the main result of this paper. The same formula with slightly different interpretations of the symbols also applies to the thermoelastic problem as we will now show.

For thermoelasticity, (2) is replaced by

$$e_{pq}^{(h)} = S_{pqrs}^{(h)}\sigma_{rs} + \beta^{(h)}\theta\delta_{pq}, \tag{9}$$

where σ is again the applied external stress in the host, $S^{(h)}$ is the compliance tensor of the host material, $\beta^{(h)}$ is the linear thermal expansion coefficient of the host material, and θ is the temperature change. Equation (3) is replaced by

$$-e_{pq}^{(h)} = \left[\frac{p_c}{3K^{(h)}} - \beta^{(h)}\theta\right]\delta_{pq}, \tag{10}$$

where $K^{(h)}$ is the bulk modulus of the host material and p_c is again the uniform confining pressure at infinity. (We do not need to distinguish types of pressure in the thermoelastic problem, so the subscript c is superfluous in this case.) Again similar expressions are obtained for the inclusion phase and for the composite medium as a whole. To ensure uniform expansion or contraction in the thermoelastic problem, we see that the ratio of pressure to temperature change must be

$$p_c/\theta = 3\,\frac{\beta^{(h)} - \beta^{(i)}}{1/K^{(h)} - 1/K^{(i)}} \equiv X, \tag{11}$$

again depending only on the physical properties of the host and inclusion phases.

The uniform strains in the composite, host, and inclusion phases when $p_c/\theta = X$ are given by

$$\varepsilon_{pq}^{*}(\theta) = -\frac{\theta}{3K^*}(X - 3\beta^* K^*)\delta_{pq}, \tag{12}$$

$$\varepsilon_{pq}^{(h)}(\theta) = -\frac{\theta}{3K^{(h)}}(X - 3\beta^{(h)} K^{(h)})\delta_{pq}, \tag{13}$$

and

$$\varepsilon_{pq}^{(i)}(\theta) = -\frac{\theta}{3K^{(i)}}(X - 3\beta^{(i)} K^{(i)})\delta_{pq}. \tag{14}$$

Again, all three of these strains (12)–(14) are equal by construction.

Now Eq. (8) can be reinterpreted for the thermoelastic single ellipsoidal inclusion problem by simply using (12) and (14) in place of (5) and (7). If a change of temperature θ occurs, then a strain e_{pq}^* imposed at infinity will result in the strain $e_{pq}^{(i)}$ in the inclusion. If the imposed strain happens to equal the one that would produce the uniform strain determined by (12)–(14), then (8) guarantees that the uniform strain outside is the same one that results inside the inclusion. If there is no change in temperature $\theta = 0$, then the terms in ε drop out of (8), and the problem reduces correctly to Eshelby's original problem.

It is worthwhile to note that (8) could have been derived in an equally rigorous, but perhaps less intuitive manner, without the use of our two thought experiments—just as Levin's derivation [18] of the thermoelastic composites' formula was obtained in a less intuitive fashion than the one of Cribb [19]. For example, the book of Mura [21] makes extensive use of the rather technical concepts of "eigenstress," "eigenstrain," and "stress-free strain," special cases of which could have been designed to permit an alternative derivation of (8) for the case of thermoelasticity. We believe, however, that the derivation presented here is much simpler, more intuitive, and easier to grasp.

The result (8) is of great practical value for many applications as mentioned earlier in the paper. For example, the result can be used in a very direct way to rederive the results of Berryman and Milton [13] and then generalize these results approximately to multicomponent porous composites using effective medium theory [8]. Another important application is the computation of long-wavelength scattering from an ellipsoidal inclusion in an infinite medium. Such results have been shown to be very useful in effective medium theories [6–8] for elastic composites. Scattering from a spherical inclusion of one poroelastic material imbedded in another has been computed previously by Berryman [22] and by Zimmerman and Stern [23], but to date no results are known for scatterers having more general shapes (such as ellipsoids) in poroelastic applications. Earlier work of Mal and Knopoff [24] writing elastic scattering formulas in terms of integral equations valid for long wavelengths has been used previously in scattering-based formulations of effective medium theory for elasticity [5,7]. By generalizing the methods of Mal and Knopoff to poroelasticity and thermoelasticity, the formulas presented here will make it possible to obtain scattering formulas for arbitrary ellipsoidal-shaped inclusions with much less effort than has been expended previously just for the spherical case, and also permit the generalization of effective medium theory [25–27] to proceed more easily into the complex realms of poroelasticity and thermoelasticity.

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