Luttinger Theorem in One Dimensional Metals

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One dimensional metals are described by Luttinger liquid theory. Recent experiments have addressed the relation between this non-Fermi liquid behavior and the existence of a Fermi surface. We show that Luttinger's theorem, with few modifications, holds for the one dimensional Tomonaga-Luttinger model. The implications for the high temperature superconductors are discussed. [S0031-9007(97)03764-2]

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Soon after the discovery of the high temperature superconductors (HTSC), Anderson [1] noted that the two dimensional (2D) copper-oxide planes in these materials are responsible for the high critical temperatures. He suggested that the essential properties of the HTSC's are contained in the physics of the 2D strongly correlated electron liquid. While there is a consensus on this question, there are debates on the nature of the low energy physics of that liquid.

The question is whether 2D quantum fluids are described by Landau's Fermi liquid theory or by a theory which resembles the physics of one dimensional (1D) systems with short range interactions [2–5]. Perturbation theory shows that despite some peculiarities compared to three dimensions [6,7], the essential physics of the 2D weakly coupled electron liquid is described by Fermi liquid quasiparticles. However, in 1D perturbation theory is known to violate some of the exact results and therefore it is feasible that in 2D the same could happen leading to unknown behavior. 1D metals with short range interactions are described by Haldane's Luttinger liquid theory [8]. In 1D, the exactly solvable Hubbard [9] and Tomonaga-Luttinger [10,11] models capture the essential physics, where instead of a simple pole, the one-particle Green's function has a branch-cut singularity and spin and charge propagate with different velocities [12,13] (the Hubbard model in the metallic phase scales (in renormalization group sense) to the Tomonaga-Luttinger model [14–16]). These properties of 1D systems are also important in understanding the properties of the organic and inorganic quasi-1D metals [17,18]. While we know that in 2D when very strong or long range interactions are present the Fermi liquid picture breaks and the system can develop charge and/or spin density waves, or condense into a Wigner crystal, the question is if this is the case for arbitrary small, short range interactions as is the case in 1D.

Some time ago, Luttinger [19] proved a theorem using perturbation theory showing that the volume enclosed by the Fermi surface is an interaction strength invariant (therefore the Fermi sphere is incompressible) and that the momentum distribution function has a discontinuity at the Fermi momentum of the noninteracting system [20]. In 1D, there is no discontinuity in the momentum distribution function at p_F, the Fermi momentum of the noninteracting system and the excitations consist only of collective boson modes. The momentum distribution function, in the vicinity of the Fermi "surface," behaves as $|k - p_F|^\alpha$ [21] with nonuniversal, coupling dependent α, and no quasiparticles are present in the liquid. The absence of quasiparticles near the Fermi surface gives the distinct properties of the 1D metals. Nevertheless, even in 1D, the Fermi momentum of the noninteracting system plays an important role when the interactions are switched on. The exact relationship between Fermi liquid behavior and the existence of a Fermi surface is not yet clear and we feel it is important to discuss this relation in Luttinger's theorem. In the case of 1D metals there have been controversial statements [22–25] about the validity of this theorem and this is a question that we address in this short paper.

In this Letter, we prove Luttinger's theorem in 1D and therefore show that the theorem holds in a system in which the interactions do not produce quasiparticles: the 1D g-ology model [21]. This theorem has been proven order by order in perturbation theory using the general properties of a Fermi liquidlike Green's function. However, in 1D there exists a closed integral equation for the single-particle Green's function obtained after the summation of the perturbation series. A particular case permits an explicit solution in k space for the Green's function which is a double-valued function of the frequency. No k space solution is available in the general case, but few results regarding the single-particle Green's function can be proven without having an explicit expression for its solutions.

Ignoring umklapp and the backward scattering processes, the g-ology model describes a set of electrons in 1D with the Hamiltonian [21]:

$$H = H_0 + H_{\text{int}}\,, \tag{1}$$

$$H_0 = \sum_{k,\sigma} v_F(k - p_F) a^\dagger_{k,\sigma} a_{k,\sigma} + \sum_{k,\sigma} v_F(-k - p_F) b^\dagger_{k,\sigma} b_{k,\sigma}\,, \tag{2}$$

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$$H_{\rm int} = \frac{1}{L}\sum_{k_1,k_2,p,\alpha,\beta}\Big[\Gamma^2_{\alpha,\beta}a^\dagger_{k_1,\alpha}b^\dagger_{k_2,\beta}b_{k_2+p,\beta}a_{k_1-p,\alpha} + \frac{1}{2}\Gamma^4_{\alpha,\beta}(a^\dagger_{k_1,\alpha}a^\dagger_{k_2,\beta}a_{k_2+p,\beta}a_{k_1-p,\alpha} + b^\dagger_{k_1,\alpha}b^\dagger_{k_2,\beta}b_{k_2+p,\beta}b_{k_1-p,\alpha})\Big], \quad (3)$$

where $a^\dagger, a$ ($b^\dagger, b$) are the creation and annihilation operators for particles at the $+p_F$ ($-p_F$) branches, respectively. $\Gamma^i_{\alpha,\beta} = g_{i\parallel}\delta_{\alpha,\beta} + g_{i\perp}\delta_{\alpha,-\beta}$ ($i = 2, 4$), and $\parallel$ and $\perp$ correspond to particles with parallel and antiparallel spins, respectively. Using a Ward identity, one obtains Dyson's equation as a singular integral equation for the single-particle Green's function for electrons moving to the right [12]:

$$G_+(p,\epsilon) = G_{0+}(p,\epsilon)\times\Big[1 + \frac{i}{4\pi^2}\int\int dk\, d\omega\, G_+ \times (p-k, \epsilon-\omega)K(k,\omega)\Big], \quad (4)$$

where

$$K(k,\omega) = \sum_{i=c,s}\Big\{\frac{A_i}{\omega - u_i k + i\delta[k]} + \frac{B_i}{\omega + u_i k - i\delta[k]}\Big\}, \quad (5)$$

and

$$G_{0+}(p,\epsilon) = \frac{1}{\epsilon - v_F(p-p_F) + i\delta[p-p_F]} \quad (6)$$

is the single particle noninteracting Green's function, and the constants A_i and B_i depend on the couplings $g_{2\parallel}$, $g_{2\perp}$ (for particles on different branches) and $g_{4\parallel}$, $g_{4\perp}$ (for particles on the same branch) [21](we assume that the couplings are momentum independent). The charge and spin velocities, u_c and u_s, respectively, are functions of the coupling constants. Here $p_F = \pi n/2$ and $\delta[q] \equiv \delta sign[q]$.

Luttinger's theorem [19,20] states that in d dimensional space (i)

$$2\int_{G(p,0)>0} dp = 2\int \theta(p-p_F)\,dp = V_F = (2\pi)^d\frac{N}{V}, \quad (7)$$

where p_F is the Fermi momentum of the noninteracting system, N is the mean number of particles in the system, V is the volume of the system, V_F is the volume of the Fermi sphere, and $G(p,0)$ is the interacting Green's function. (ii) The momentum distribution function $n(\mathbf{p})$ has a discontinuity at the points $\{\mathbf{p}_F\}$, at which the noninteracting distribution function $n_0(\mathbf{p}) \sim \theta(\mathbf{p}-\mathbf{p}_F)$ is discontinuous. The discontinuity of $n(\mathbf{p})$ is proportional to the quasiparticle residue, i.e., $\lim_{\mathbf{p}\to\mathbf{p}_F}[n(\mathbf{p}<\mathbf{p}_F) - n(\mathbf{p}>\mathbf{p}_F)] = Z$. However, in 1D Z is zero and the generalized statement is that the derivative of $n(\mathbf{p})$ is singular with a power law singularity instead of a delta function singularity. We will show that Luttinger's theorem is satisfied (in 1D) by the Green's function satisfying Eq. (4), i.e., that it changes sign when crossing the Fermi momentum of the noninteracting system. In this case the interacting Green's function is a product of [see Eq. (4)] the noninteracting Green's function, which changes sign at p_F and a term which, if it does not change sign when we cross p_F, Eq. (7) and therefore the first part of Luttinger's theorem will be satisfied. From now on we work with particles on the right branch, i.e., moving to the right. Let us denote by $D(p)$ the term in the parentheses at $\epsilon = 0$, i.e.,

$$D(p) = \frac{i}{4\pi^2}\int\int dk\, d\omega\, G_+(p-k,-\omega)K(k,\omega). \quad (8)$$

We can integrate (we explain later in the paper in more detail how these integrals are calculated) over ω using the general properties of fermionic Green's function [26]: (i) The singularities are located in the second quadrant in the complex-frequency plane for $p - p_F < 0$ and in the fourth quadrant for $p - p_F > 0$; (ii) $G(p,\omega) \overset{\omega\to\infty}{\to} \frac{1}{\omega}$. The result for $p > p_F$ is

$$\frac{1}{2\pi}\sum_{i=c,s}\Big[B_i\int_{-\infty}^{0} G_+(p-k,u_i k)\,dk - B_i\int_0^\infty G_+(p-k,u_i k)\theta(p_F-p+k)\,dk + A_i\int_0^\infty G_+(p-k,-u_i k)\theta(p-k-p_F)\,dk\Big] \quad (9)$$

and for $p < p_F$

$$\frac{1}{2\pi}\sum_{i=c,s}\Big[B_i\int_{-\infty}^{0} G_+(p-k,u_i k)\theta(p-k-p_F)\,dk - B_i\int_0^\infty G_+(p-k,u_i k)\,dk - A_i\int_{-\infty}^{0} G_+(p-k,-u_i k)\theta(p_F-p+k)\,dk\Big] \quad (10)$$

In the limit $p \to p_F$ the last two expressions are equal and the limit is

$$\lim_{p\to p_F} D(p) = \frac{1}{2\pi}\sum_{i=c,s}B_i\Big[\int_{-\infty}^{0} G_+(p_F-k,u_i k)\,dk - \int_0^\infty G_+(p_F-k,u_i k)\,dk\Big] = -1 \quad (11)$$

above and below p_F. Comparison of the exact expression for the momentum distribution function and its expansion

around p_F (both shown later) shows that

$$D(p \cong p_F) = -1 + \text{const} \times |p - p_F|^{\alpha} + \text{const} \times (p - p_F). \quad (12)$$

Therefore

$$G_+(p \cong p_F, 0) \sim -\text{const} \times \frac{|p - p_F|^{\alpha}}{p - p_F} \quad (13)$$

and therefore the Green's function changes sign at $p = p_F$ which completes the proof of the first part of Luttinger's theorem.

Next, we show that the Green's function given by Eq. (4) gives the same number of particles as the noninteracting one. We use a band width cutoff A. Practically, that means that when the Green's function is integrated over the momentum, one integrates in the interval $[p_F - A, p_F + A]$ and then takes the limit $\Lambda/p_F \to 0$, where $\Lambda = p_F - A$. The number of particles will be the same if and only if

$$\frac{1}{4\pi^2} \lim_{t\to 0^+} \int\int \frac{dp\, d\epsilon}{(2\pi)^2} G_{0+}(p,\epsilon) e^{i\epsilon t} \times \int\int dk\, d\omega\, G_+(p - k, \epsilon - \omega) K(k,\omega) = 0 \quad (14)$$

since

$$-2i \lim_{t\to 0^+} \int\int \frac{dp\, d\epsilon}{(2\pi)^2} G_{0+}(p,\epsilon) e^{i\epsilon t} = \frac{p_F}{\pi}\left(1 - \frac{\Lambda}{p_F}\right) = n_+ . \quad (15)$$

Here n_+ is the density of particles moving to the right.

Using the mentioned properties of the Green's function, first we integrate over the frequencies ϵ. Because of the second property, the integrals on the left-hand side of Eq. (14) are convergent and therefore one can introduce the limit under the integral and perform it explicitly before integrating. This allows us to close the contour of integration on either side of the real frequency axis in the complex plane. Therefore, when the branch-cut singularity of the Green's function and the pole occur on the same half plane we close the contour in the other half plane and from the Cauchy theorem the corresponding integral is zero. When the pole and the branch-cut singularity of the Green's function are on the opposite side of the real axis we choose to close the contour of integration in the half complex plane where the pole is located and therefore we obtain the branch-cut part evaluated at that pole. Then we evaluate the second frequency integration in (14) using the same procedure. After performing the double frequency integration, the A_i terms give zero both in the charge and spin sectors. This is the reason why the simple square root Green's function [12] leads to the step-function momentum distribution function, the same as for the noninteracting case. However, the double frequency integration of the B_i terms is nonzero and the result is

$$n_+(p > p_F) = \frac{1}{2\pi} \sum_{i=c,s} B_i \int G_+(p - k, v_F(p - p_F) + u_i k)\theta(p_F - p + k)\theta(k)\, dk$$
$$n_+(p < p_F) = 1 + \frac{1}{2\pi} \sum_{i=c,s} B_i \int G_+(p - k, v_F(p - p_F) + u_i k)\theta(p - k - p_F)\theta(-k)\, dk . \quad (16)$$

The exact value of the last two integrals cannot be obtained without an explicit expression for the function. However, from the exact solution [27] and from perturbative calculations [12,21,28,29], it is known that the momentum distribution function is continuous and without a jump at $p = p_F$ and for $p \simeq p_F$ is

$$n(p > p_F) \sim \frac{1}{2} - C_1 |p - p_F|^{\alpha} - C_2(p - p_F),$$
$$n(p < p_F) \sim \frac{1}{2} + C_1 |p - p_F|^{\alpha} - C_2(p_F), \quad (17)$$

where C_1 and C_2 are constants [28]. From the last two representations of the momentum distribution function follows the behavior of the integrals of the type encountered in the expression for $G_+(p,0)$ in the neighborhood of p_F.

In the perturbation theory accessible regime ($\alpha < 1$) the derivative $\frac{dn}{dp} \sim |p - p_F|^{\alpha - 1}$ is singular approaching infinity with a power law. We adopt the general definition that the Fermi surface is the set of k points at which the mth derivative of the momentum distribution function has a singularity, i.e., $\{p_F\} \equiv \{\forall\, k\colon \frac{d^m n(k)}{dk^m} \text{ is singular}\}$. These we shall call Fermi points of order m. In the usual Fermi liquid the Fermi surface consists of zero order Fermi points while in the Luttinger liquid these are of first order. Although the zero order Fermi surface has disappeared, the first order Fermi surface is left and the generalized statement of the second part of Luttinger's theorem holds. When $\alpha > 1$ liquid droplets form and as long as α is not an integer the derivative is zero at p_F, but there will exist a number m so that $\frac{d^m n}{dp^m}$ is singular at $p_F$and this will correspond to a Fermi surface of order m.

At the end we must show that the following expression is zero in the limit $\Lambda \to 0$:

$$I = \frac{1}{2\pi} \sum_{i=c,s} B_i \Bigg[\int_{\Lambda}^{p_F} dp \int_{-\infty}^{p-p_F} dq\, G_+(q, v_F(p - p_F) + u_i(p - q)) + \int_{p_F}^{2p_F - \Lambda} dp \int_{p-p_F}^{\infty} dq\, G_+(q, v_F(p - p_F) + u_i(p - q)) \Bigg]. \quad (18)$$

Taking into account the two representations of the momentum distribution function one sees that in the above mentioned limit the two integrals cancel each other and the total number of particles on the right branch is $n_+ = p_F/\pi$.

In this paper, we have shown from Dyson's equation, that Luttinger's theorem holds for the one dimensional Tomonaga-Luttinger model. In general, the theorem is based on the counting of the fermionic degrees of freedom before and after the interactions are turned on. In the case of Fermi liquids, the one-to-one correspondence between the noninteracting particles and the quasiparticles ensures the validity of the theorem. In the case of 1D Luttinger liquids, the number of charge particles in the interacting system is exactly equal to the number of electrons in the noninteracting system. Our conjecture is that as long as the number of states and excitations with and without interactions are the same, Luttinger's theorem will be satisfied. Some recently proposed 2D Luttinger-type Green's functions satisfy this condition [4,30]. The HTSC's, at zero doping and below the Néel temperature, are antiferromagnetic insulators. Doping destroys the antiferromagnetic order and a metallic phase occurs above the superconducting critical temperature. Our discussion indicates that in the high temperature superconducting cuprates, Luttinger's theorem will be satisfied in this temperature and doping interval regardless of the nature of the electronic liquid (Fermi or Luttinger).

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