Noise and Fluctuations of Rough Surfaces

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Correlated non-Gaussian noise of the quenched disorder on rough surfaces is analyzed on the basis of stochastic differential and integral equations. The irregular fluctuations reflect the microstructure of rough surfaces and the stochastic process is treated as Brownian motion. The microstructure refers to the standard deviation σ, correlation length ξ, and roughness exponent α that defines the scaling properties of the surface. We have derived the noise correlation function $\langle \eta(0)\eta(r)\rangle \sim (r/\xi)^{2\alpha-2}$ for $1/2 < \alpha < 1$ on a self-affine rough surface. This provides a physical justification of noise with long-range correlation, and the anomalous behavior of $\alpha > 1/2$ observed experimentally. [S0031-9007(97)03827-1]

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Rough surfaces are the main focus of a great deal of recent research on the stochastic growth and fractal geometry of the surface. These are the topics of interesting and timely reviews [1–3] where extensive references can be found. On a rough surface or interface, the quenched noise is generated by disorder and does not change with time. It is much more important than the thermal noise that is always present. In the cases when the statistics of the noise is uncorrelated Gaussian, as is almost always assumed [1], the microscopic details are not important on the large scales. This is known as universality and the corresponding α has the well-known value of $\frac{1}{2}$.

Experimental results, however, have revealed that many rough surfaces, though self-affine ($\alpha < 1$), behave anomalously with $\alpha \approx 0.6$–0.8 [2,4–6] that exceeds $\alpha = \frac{1}{2}$. Additional references can be found in section 4 of [2]. Therefore, it is reasonable to expect that the microstructure of rough surfaces can indeed influence the noise correlation function, which violates the concept of universality. In order to explain this observed anomaly, alternative physical ideas like correlated noise, non-Gaussian, and power-law noise have been suggested [2,3,7–11]. Perhaps the earliest study of this anomaly involving the spatially correlated Gaussian noise was reported by Medina *et al.* [7]. They made approximate predictions of scaling exponents, but did not quite resolve the problem [2]. Zhang [8] proposed that the amplitude of noise follows a power-law distribution, and showed that the roughness exponent may change even in the presence of uncorrelated noise. Despite the interest in this power law as a possible mechanism to account for $\alpha > \frac{1}{2}$, severe obstacles remain, because the power law is solely on mathematical expediency with essentially no physical justification [2,8].

In this paper, the irregular fluctuations of a rough surface will be analyzed as a stochastic process expressed in terms of Brownian motion. We would like (1) to seek a better understanding of the long-range noise correlation, and (2) to derive an explicit expression that links the noise correlation function $\langle \eta(0)\eta(\vec{r})\rangle$ to the microstructure (σ, ξ, α) of rough surfaces in the discussion of long tails. In a disordered medium, the quenched noise generated by the disorder is usually more important than that of temporal noise [2,3,12]. Through this Letter, we shall focus on surface roughening affected by quenched disorder which does not change with time.

The height of a continuous rough surface from its smooth reference is represented by the function $h(\vec{r})$, where $\vec{r}$ is the position vector on the reference surface. It is usual to ensure that $\langle h\rangle = 0$ where the angular bracket denotes the average across the surface. Three independent parameters are needed to describe the microstructure of a rough surface. The standard deviation σ is the root mean square fluctuation normal to the surface, and the correlation length ξ parallel to the surface. In addition to the length scales, the third independent parameter is the roughness exponent $\alpha = d - d_f$ where d is the spatial dimension and d_f is the local fractal dimension. The roughness exponent defines the scaling properties of the surface and is equal to $\alpha = \frac{1}{2}$ in the case of uncorrelated Gaussian noise.

For a self-affine rough surface, the change of the height correlation function with distance r is given by

$$C(\vec{r}) \equiv \langle [h(\vec{r}_0 + \vec{r}) - h(\vec{r}_0)]^2\rangle \sim r^{2\alpha} \quad \text{for } r \ll \xi \,. \tag{1}$$

At long range, the global behavior is described by $C(\vec{r}) = \sigma^2$ for $r \gg \xi$. The correlation length ξ can be defined by a correlation of fluctuations of $h(\vec{r})$ at two points $\vec{r}_0$ and $\vec{r}_0 + \vec{r}$ [13]:

$$\psi(\vec{r}) \equiv \langle h(\vec{r}_0)h(\vec{r}_0 + \vec{r})\rangle - \langle h(\vec{r}_0)\rangle^2 = \sigma^2 - C(\vec{r})\,, \tag{2}$$

which goes to zero when the two heights become uncorrelated at the distance of the order of ξ. Therefore, $\xi = \int r\psi\, dr / \int \psi\, dr$.

An exact solution of the height correlation function in the entire range of r can be determined by the

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following stochastic differential equation which describes the fluctuations of local slope on a rough surface:

$$\frac{d(\Delta h)}{dr} = -\frac{\Delta h}{2\xi} + \eta(r), \qquad (3)$$

where $\Delta h(\vec{r}) = h(\vec{r}) - h(\vec{0})$. The first term on the right hand side is the average local slope, and η is the noise term that is the source of fluctuations of Δh. It has zero average. Differing from the uncorrelated noise [8], the nature of noise in this Letter is described by a correlation function. Our main purpose is to derive a nontrivial average of quenched noise along a given interface. Equation (3) has the form of the Langevin equation for Brownian motion, whose concepts and methods are applicable to a wide class of physical phenomena. In this paper, the velocity of a Brownian particle [14] is replaced by Δh, the variable time by r, and the frictional coefficient by $(2\xi)^{-1}$. Integrating Eq. (3), squaring it, and taking the mean, we get

$$\langle[\Delta h(r)]^2\rangle = \exp(-r/\xi) \times \int_0^r \int_0^r \exp[(r_1 + r_2)/2\xi] \times \langle\eta(r_1)\eta(r_2)\rangle\, dr_1\, dr_2 . \qquad (4)$$

Our main interest is the noise correlation function. In cases when the statistics of noise is Gaussian, as is assumed in most applications [1], one has the uncorrelated white noise with $\langle\eta\rangle = 0$ and

$$G(r_1 - r_2) \equiv \langle\eta(r_1)\eta(r_2)\rangle = A\delta(r_1 - r_2) . \qquad (5)$$

The constant A is determined by the requirement of $\langle\Delta h^2\rangle = \sigma^2$. This gives $A = \sigma^2/\xi^2$. Substituting Eq. (5) into Eq. (4) yields

$$C(r) = \sigma^2[1 - \exp(-r/\xi)] = \sigma^2/\xi + \ldots, \quad \text{for } r \ll \xi . \qquad (6)$$

Comparing Eqs. (1) and (6), we obtain (1) $\alpha = \frac{1}{2}$, and (2) a generalization of Eq. (6) to the cases of $\alpha \neq \frac{1}{2}$ as

$$C(r) = \sigma^2\{1 - \exp[-(r/\xi)^{2\alpha}]\} . \qquad (7)$$

A straightforward mathematical deduction is used in the derivation of Eq. (7). We have exponentiated an infinite series similar to that of Eq. (6). The infinite series $1 - (r/\xi)^{2\alpha} + \ldots$ is uniformly convergent in $0 \leq r/\xi \leq 1$. At the same time, the roughness exponent α happens to be significant only within the domain of convergence (see Fig. 7.4 in [13]). A physical justification has now been obtained for Eq. (7) whose form was previously proposed but without explanation [13,15]. Equation (7) not only gives the exact limits of $C(r)$ when r is either much greater or smaller than ξ as shown in Eq. (1), but is also in good agreement with the numerical result based on computer simulations [3]. It covers the crossover region in the vicinity of $r/\xi = 1$ between the above mentioned limits.

Spectral distribution is often used to discuss the observed surface topography [16]. An important measure of the surface statistics is the autocorrelation function $\psi(r)$ that is real. For stationary surfaces the autocorrelation function can be expressed in terms of the power spectral density $\overline{\psi}(q)$ by a Fourier transform:

$$\overline{\psi}(q) = \int_{-\infty}^{\infty} \psi(r)\exp(-iqr)\, dr = 2\,\mathrm{Re} \int_0^{\infty} \psi(r)\exp(-iqr)\, dr , \qquad (8)$$

where q is the spatial frequency of the undulations on the surface. The function $\psi(r)$ is given by the second term in Eq. (7) in accordance with Eq. (2). Two special cases can be calculated analytically:

$$\overline{\psi}(q) = \frac{2\sigma^2\xi}{1 + (q\xi)^2}, \quad \text{for } \alpha = \frac{1}{2}, \qquad (9)$$

and

$$\overline{\psi}(q) = \sqrt{\pi}\sigma^2\xi\, \exp[-(q\xi)^2/4], \quad \text{for } \alpha = 1 . \qquad (10)$$

These two functions are illustrated in Fig. 1, which provides us with the essential information about the noise and fluctuations of rough surfaces. Chaotic behavior is marked by a broad band of continuous power spectrum [17]. In Fig. 1, we see such broad spectrum and the relatively uninformative region $q\xi < 1$. Therefore, we shall focus our attention to the region $q\xi > 1$ in the following study of correlated noise.

Each Langevin equation has a corresponding Fokker-Planck equation. From this we can derive an integral equation that establishes the relation between the autocorrelation function $\psi(r)$ and the noise correlation function $G(r)$ [14]:

$$\frac{d\psi(r)}{dr} = -\frac{1}{\sigma^2}\int_0^r G(r - s)\psi(s)\, ds . \qquad (11)$$

This integral equation follows directly from a statistical average of the Fokker-Planck equation for Brownian motion, and it has automatically taken into account the non-Gaussian memory effect. Replacing $iq = p/\xi$ in

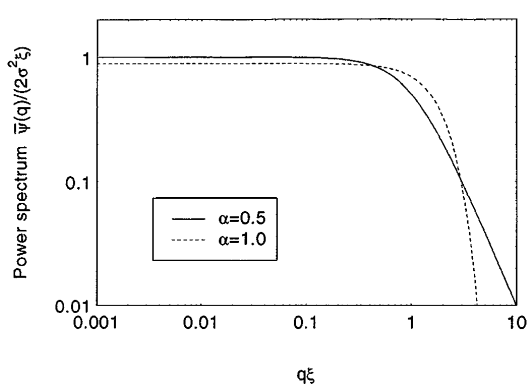

FIG. 1. The surface spectral power $\overline{\psi}(q)$ versus the nondimensional spatial frequency $q\xi$ of rough surfaces.

Eq. (8), we consider the Laplace transform of Eq. (11):

$$\overline{G}(p) = \int_0^\infty G(u)\exp(-pu)\,du = -A\,\frac{p\overline{\psi}(p) - \sigma^2}{\overline{\psi}(p)} \quad \text{with } A = \sigma^2/\xi^2. \tag{12}$$

According to a mathematical theorem [18]

$$\lim_{p\to\infty} p\overline{\psi}(p) = \lim_{u\to 0}\psi(u) \tag{13}$$

we write the series expansion of the autocorrelation function:

$$\psi(u)/\sigma^2 = \exp(-u^{2\alpha}) = 1 - u^{2\alpha} + \frac{u^{4\alpha}}{2!} - \frac{u^{6\alpha}}{3!} + \dots, \quad \text{with } u = r/\xi\,. \tag{14}$$

Equations (11) and (14) clearly reveal the non-Gaussian characteristics of a non-Markovian process [19]. Taking the Laplace transform of Eq. (14) and then substituting it into Eq. (12), we obtain

$$\overline{G}(p) = -A\,\frac{p\sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(2\alpha n+1)}{n!p^{2\alpha n}}}{1 + \sum_{n=1}^{\infty}(-1)^n\frac{\Gamma(2\alpha n+1)}{n!p^{2\alpha n}}}\,, \tag{15}$$

where Γ is the gamma function. When $\alpha = \frac{1}{2}$, we get $\overline{G}(p)/A = 1$ whose Laplace inversion is the delta function mentioned in Eq. (5). The Laplace inversion of the leading term in Eq. (15) gives

$$\langle\eta(0)\eta(r)\rangle = \frac{\sigma^2}{\xi^2}\,\frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha - 1)}\left(\frac{r}{\xi}\right)^{2\alpha-2} \quad \text{for } \frac{1}{2} < \alpha < 1 \tag{16}$$

in the limit of $r/\xi \ll 1$. Of course, Eq. (15) is needed for $0 \le r/\xi < 1$. Equation (16) is the sought after equation for the long-range noise correlation in space which is ubiquitous in nature. Interestingly, the long tail is directly related to the roughness exponent and correlation length. The strength of the noise correlation depends also on the standard deviation σ. Our theory is good for the anomalous behavior of rough interfaces observed experimentally with the roughness exponent $\alpha \approx 0.6$–0.8 [2,4–6] that exceeds $\alpha = \frac{1}{2}$. An uncorrelated power-law noise has already been reported [8] as a possible mechanism for the above mentioned experiments of anomalous interfaces ($\alpha > \frac{1}{2}$), and has received a great deal of interest. However, one of its severe obstacles is that this power law is based solely on mathematics without physical justification [2].

The present theory will be useful for a better understanding of the important effect of quenched disorder on interesting technical problems like wetting [12], adhesion, and diffuse scattering [20]. Consider the wetting of rough surfaces. The contact angle θ of a liquid drop on a rough solid can be related to a force F per unit length on the contact line by

$$\gamma(\cos\theta - \cos\theta_0) = F - F_c\,, \tag{17}$$

where γ is the interfacial tension of liquid-vapor interface, θ_0 is Young's contact angle, and F_c is the critical depinning force per unit length. Quenched disorder plays an important role in the depinning transition [2,3]. Our newly derived correlation functions suggest that roughness enhances wetting. Further advances in this area are needed and can be benefited by good experiments that relate macroscopic behavior (θ) to microstructure (σ, ξ, α). This could be another example showing the essence of anomalous interface that exhibits the long-range noise correlation.

In summary, analytical expressions for the noise correlation function, Eqs. (15) and (16), and the height correlation function, Eq. (7), have been derived as a function of the quenched microstructure (σ, ξ, α) of a rough surface. From the calculated behavior of surface spectral power, we see that the correlated noise should occur in the region $q\xi > 1$. On the basis of the stochastic differential and integral equations, a better understanding of the physics behind the noise with long-range correlation is obtained. The long tail is a result of the non-Markovian fluctuations on a rough surface which carries a memory effect. In addition, we have derived that $\alpha > \frac{1}{2}$ for the correlated non-Gaussian noise on a self-affine rough surface. This provides a physical justification of the observed anomalous roughness exponent. Furthermore, our theory has the potential to improve the understanding of the wetting of rough surfaces.

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