Lyapunov Exponents, Singularities, and a Riddling Bifurcation

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There are few examples in dynamical systems theory which lend themselves to exact computations of macroscopic variables of interest. One such variable is the Lyapunov exponent which measures the average attraction of an invariant set. This article presents a family of noninvertible transformations of the plane for which such computations are possible. This model sheds additional insight into the notion of what it can mean for an attracting invariant set to have a riddled basin of attraction. [S0031-9007(97)03805-2]

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Dynamical systems theory studies the evolution of a given system. While many studies concentrate on quantifying the long-term behavior of invertible maps, not all dynamical systems have this property; separate points may map to a single phase point after one or many iterations, while other phase points may not have a forward image at all. In this latter case, these maps are defined as noninvertible.

The primary focus of this article is a riddling bifurcation found in families of noninvertible maps and the role played by singularities in their global dynamics. It is likely that such dynamics are always present whenever Newton's method is used to find the stationary solutions to an evolution equation.

Since complicated dynamical behavior can often come from the simplest maps, we will present examples of this riddling bifurcation in systems derived from several families of polynomial factorization methods applied to low-order polynomials. By studying the interactions of fixed points, singular curves, and invariant lines, we present a possible new bifurcation, or "eruption." We define an eruption in a noninvertible mapping as a bifurcation involving the merger of an attracting periodic orbit or fixed point with a point on a singular curve. This results in a transfer of stability from the attracting periodic orbit to another invariant set. For a more complete discussion of eruptions, see the recent publication by Billings and Curry [1].

Boyd noted in his 1977 study of Bairstow's factorization method [2] that, for a "modification" of that method, an invariant line exhibited a "type of stability." That is, an initial condition chosen in a neighborhood of the invariant line appeared to remain in a neighborhood of that line, but not in the normal way of shadowing. We believe that Boyd may have discovered an early example of transverse stability. He also reported that points escaped from a neighborhood of the invariant line and, hence, may have also discovered an occurrence of what is now known as riddling of the basin of attraction of the invariant line.

While noninvertible dynamical systems in greater than one dimension have not been widely studied, several recent articles that explored this area involved work by Lai, Grebogi, Yorke, and Venkataramani [3], Alexander, Hunt, Kan, and Yorke [4], and Bischi and Gardini [5].

The context of our problem is as follows: As a parameter a increases through the critical value a_c, two attracting fixed points merge along the invariant line into the point where the singular curve intersects it.

Definition 1 (singular curve)—A point (u, v) is singular if, in a rational map, the denominator of any component is zero at the point (u, v). A set S is singular if every $(u, v) \in S$ is singular. A singular curve is a singular set defined by a function $h(u, v) = 0$.

Definition 2 (invariant line)—A set S is invariant for a function f if, for all $x \in S$, $f(x) \in S$. An invariant line is a line with this invariance property.

For $a > a_c$, the invariant line persists but with infinitely many periodic orbits and no fixed points. The dynamics along the invariant line are expanding, and the Lyapunov exponent corresponding to the eigenvector parallel to the line is greater than 0. The Lyapunov exponent for the transverse eigenvector is negative for $a_c < a < a_0$, where we define the parameter a_0 to be that parameter value for which the transverse Lyapunov exponent is exactly zero.

When the transverse Lyapunov exponent is negative, the invariant line is an attractor in the sense of Milnor [6]. But, due to a new dynamical phenomenon called a "focal point" [5] and an associated "bow tie," points arbitrarily close to the invariant line are allowed to escape to the basin of another attractor. The dynamics can also admit riddled basins. In the presence of singularities, we conjecture that basins must always be riddled.

Definition 3 (focal point)—A focal point for a map T is a point (u_f, v_f) such that at least one component of $T(u_f, v_f)$ maps to the form $\frac{0}{0}$.

Definition 4 (bow tie)—A bow tie in two dimensions is roughly an hourglass shaped region symmetrically divided by the focal point.

Due to noninvertibility, the preimages of the focal point are dense in the invariant line, making the preimages of the vertex of the bow tie also dense. Therefore, points

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belonging to preimages of the bow tie, no matter how close to the attracting invariant line, will escape to the other attractor.

For example, the following family of noninvertible transformations arise from factoring a cubic polynomial:

$$B_a\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{u^3+u(v-a+1)+a}{2u^2+v} \\ \frac{v(u^2+a-1)+2au}{2u^2+v} \end{pmatrix}.$$

This is Bairstow's method applied to a cubic polynomial. (See Ref. [1] for the derivation.) One fixed point has the formula $r_1 = \binom{1}{a}$. Note that the other roots, r_2 and r_3, are real for values $a \leq \frac{1}{4}$. The stability of the fixed points are determined by examining the eigenvalues of the Jacobian matrix of B_a. An immediate conclusion is that the fixed points of B_a are contractive when they exist since all entries of the Jacobian matrix are identically zero when evaluated at such points, except when $a = \frac{1}{4}$.

The singular set for B_a is defined by $\{(u, v) : 2u^2 + v = 0\}$. The focal points of B_a must belong to this set, and are $\{(1, -2), (0, 0)\}$. Only $(1, -2)$ lies along the invariant line, $L_1 : v = -u - 1$, and plays a part in the bifurcation.

The presence of an invariant line in this example is traceable to the existence of a linear factor in the underlying polynomial. (See Ref. [1] or [2] for more details.) For values of $a < 1/4$, the three fixed points are connected by three invariant lines. At $a = a_c = 1/4$, two fixed points, r_2 and r_3, and the two invariant lines connecting them to r_1 merge into one. For $a > 1/4$, only r_1 and L_1 exist.

The bow tie can be roughly estimated by two triangles with a common vertex at the focal point. The base of the triangles are parallel and equidistant from the invariant line, while the sides are defined by the tangent to the singular curve at the focal point and the tangent to the preimage of the singular curve at the focal point.

Linearizing the map in a neighborhood of the invariant line and making the substitution $v \to -u - 1$, we find that the eigenvalues and eigenvectors of the linearization at any point u along the invariant line are

$$\lambda_{\|}(u) = \frac{2(a + u + u^2)}{(2u + 1)^2}, \qquad e_{\|} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\lambda_{\perp}(u) = \frac{a + u + u^2}{(2u + 1)(u - 1)}, \qquad e_{\perp} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

As a exceeds $\frac{1}{4}$, there is an eruption, which produces an infinite number of periodic and aperiodic orbits. This can be proven by establishing the existence of a topological conjugacy between B_a restricted to the invariant line and a degree two rational mapping.

Denote the u component of B_a restricted to the invariant line, by $L_1 : \mathbf{R} \to \mathbf{R}$,

$$L_1(u) = \frac{u^2 - a}{2u + 1}, \qquad \left(a > \frac{1}{4}\right).$$

The conjugating transformation is

$$h(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{2x + 1}{\sqrt{4a - 1}}\right), \tag{1}$$

for the conjugacy: $h \circ L_1 \circ h(x)^{-1} = g(x)$, with $g(x)$ defined for $x \in I : [0, 1]$ by

$$g(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}) \\ 2x - 1, & x \in [\frac{1}{2}, 1] \end{cases}. \tag{2}$$

Note that $g(x)$ is conjugate to $L_1(u)$ and is independent of a. The dynamics of $g(x)$ are well understood from both a topological and measure theoretical point of view. Further, as long as the derivative of g exceeds one in absolute value, a result due to Bowen [7] allows us to conclude that there is an invariant ergodic measure associated with the transformed family of mappings. Using the conjugacy $h(x)$ from (1), the conjugate measure must be $\rho(dx) = \frac{dh}{dx}\, dx$. Further, preimages of the focal point are dense in the unit interval.

There are few examples where dynamical systems depending continuously on a bifurcation parameter show continuity properties in their characteristic exponents as functions of that parameter (see Ruelle [8]). Since we have explicit formulas for both the transversal and parallel growth rates associated with the linearization along the invariant line L_1, we proceed in our investigation by determining the behavior of those two eigenvalues as a is varied. Using the space average formula, we computed the Lyapunov exponents exactly:

$$\Lambda(\lambda) = \int_{x_0}^{x_1} \ln \left| \frac{df}{dx} \right| \rho(dx).$$

Then the Lyapunov exponent for B_a along the invariant line can be determined by

$$\Lambda(\lambda_{\|}) = \frac{\sqrt{4a - 1}}{2\pi} \int_{-\infty}^{\infty} \frac{\ln\left|\frac{2(a+x+x^2)}{(2x+1)^2}\right|}{x^2 + x + a}\, dx = \ln 2, \tag{3}$$

$$\Lambda(\lambda_{\perp}) = \frac{\sqrt{4a - 1}}{2\pi} \int_{-\infty}^{\infty} \frac{\ln\left|\frac{a+x+x^2}{(2x+1)(x-1)}\right|}{x^2 + x + a}\, dx = \frac{1}{2} \ln\left(\frac{4a - 1}{2 + a}\right). \tag{4}$$

The characteristic exponent associated with the rate of expansion along the invariant line agrees with our proposition that, in one dimension, the map behaves like $g(x)$ in (2). That value is constant and equal to $\ln 2$. The transversal growth rate along a typical orbit is monotonically increasing. This equation indicates a decrease in the stability of the invariant line to transverse perturbations as a increases (noted by Boyd). There is a critical parameter value at $a = 1$ for which the invariant line is, on average, neutrally stable to transverse perturbations. Our conclusion is that basin riddling for this example is initially associated with singularities, focal points, bow ties, and transverse Lyapunov exponents.

In Fig. 1, we show the basin map for the bow tie when $a = 0.75$. Notice how the white points, which are essentially preimages of the bow tie, match with the white points that converge to the fixed point in Fig. 2. These points riddle the invariant line's basin. We know that the line $u = 1$, which passes through the bow tie and the focal point, collapses to the fixed point in one step. This is more evidence why the bow tie is a region where points should escape from the attractor.

Further, the loss of stability to transverse perturbations does not imply that saddles no longer exist along the invariant line. For example, we check that the period two orbit, defined by the formula

$$\mathrm{Per2}(a) = -\frac{1}{2} \pm \frac{\sqrt{4a - 1}}{2\sqrt{3}},$$

has eigenvalues $\lambda_{\parallel} = 4$ and $\lambda_{\perp} = \frac{-1+4a}{a-7}$. This periodic point does not become a source until $a = 1.40$, well above the threshold value indicated for neutral stability of the invariant line. However, as a exceeds 1, $\Lambda(\lambda_{\perp})$ is positive, and funnels appear as described in [3].

After this discussion for B_a, the basin map in Fig. 2 makes more sense for $a = 0.75$. The basin of attraction of the invariant line is "riddled" [9] because any neighborhood containing points which converge to the invariant line also contain points that converge to the fixed point. We also note that, as the bifurcation parameter increases, the density of the points converging to the invariant line appears to decrease, as predicted by the Lyapunov exponents.

We conjecture that a consequence of having a negative Lyapunov exponent is that, for all parameter values $\frac{1}{4} < a < 1$, the invariant line must attract a set of positive measure in the plane. Such a result would be similar to the ergodic attractors theorem for C^2 functions [4]. Numerical evidence for this appears in the attractor's dimension. The box-counting dimension is very close to 2 for the range of a when the Lyapunov exponent is negative (see Table I).

Riddling in the above example is associated with the existence of a singular curve and a focal point of the map, and apparently not due to a symmetry breaking bifurcation. Eruptions are present in other factorization methods that have even more complicated dynamical behavior than B_a. Consider the map,

$$M_a\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{(1-a)(u^2-v^2)+u^4/2-\sqrt{6}\,av-2u^2v^2+v^4/6}{(u^3-3uv^2)} \\ \frac{3\sqrt{6}\,a+6(a-1)v+3u^2v-5v^3}{3(u^2-3v^2)} \end{pmatrix}.$$

This map has three invariant lines, forming an equilateral triangle. The three singular lines crossing at $(0,0)$ are the perpendicular bisectors of the triangles. For $a < \frac{1}{4}$, there are six fixed points. At $a = \frac{1}{4}$, the fixed points coalesce by pairs along their respective invariant lines into points where the singular lines cross the invariant lines, and three eruptions happen simultaneously. The focal points are at the vertices of the triangle formed by invariant lines.

There is a threefold symmetry in M_a, and the map is repeated every $2\pi/3$ radians (see Fig. 3). In each subregion, there is also a reflection symmetry splitting the region in half along the singular line contained in that region. The rest of the discussion will pertain

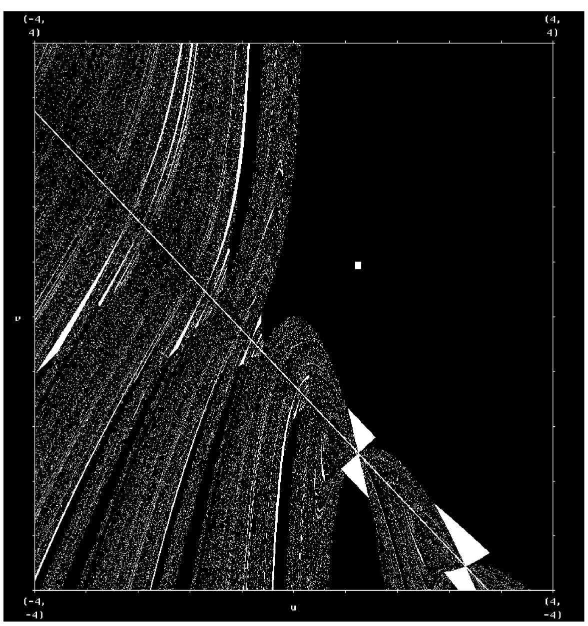

FIG. 1. This is a basin map for the bow tie phenomena in the $B_{0.75}$ map. The white points have been mapped to the bow tie within 200 iterations.

FIG. 2. This graph is the basin map associated with $B_{0.75}$. The black points have not converged to the fixed point $(1, 0.75)$ (in black) in 200 iterations.

TABLE I. Table of box-counting dimensions for the attractor in B_a as a is varied from 0.255 to 1.0.

a	Box-counting dimension
0.255	1.98846
0.3	1.98462
0.4	1.96284
0.5	1.96028
0.6	1.95588
0.7	1.93677
0.8	1.91873
0.9	1.92132
1.0	1.91078

only to $l_1(u): v = -\sqrt{3/2}$, since the dynamics in a neighborhood of other lines is similar.

We can simplify M_a to the following one-dimensional map on l_1:

$$t_1(u) = u - \frac{2u^2 + 4a - 1}{4u}.$$

We can determine a similar conjugacy to $g(x)$ defined in (2) using the function

$$j(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\sqrt{\frac{2}{4a - 1}}\, x\right). \tag{5}$$

The behavior along l_1 must be qualitatively similar to the behavior of the dynamics on the invariant line in B_a, and we expect the same Lyapunov exponent for the parallel eigenvector. But what about the transverse expansion rate?

The eigenvalues of the linearization for l_1 are

$$\lambda_\perp = \frac{2u^2 + 4a - 1}{2u^2 - 9},$$

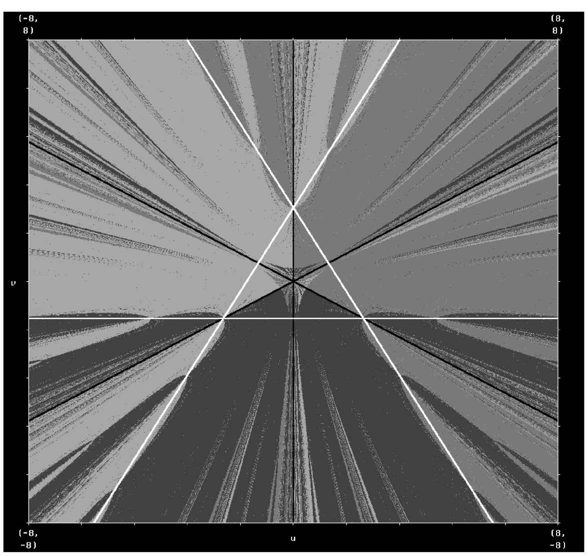

FIG. 3. This is the basin map associated with $M_{0.5}$. The three invariant lines form a triangle and the singular lines cross through the origin.

$$\lambda_\parallel = \frac{2u^2 + 4a - 1}{4u^2}.$$

As expected from the conjugacy, $\Lambda(\lambda_\parallel) = \ln 2$. The transverse Lyapunov exponent is different from (4) by a constant,

$$\Lambda(\lambda_\perp) = \frac{\sqrt{2(4a - 1)}}{\pi} \int_{-\infty}^{\infty} \frac{\ln\left|\frac{2u^2+4a-1}{2u^2-9}\right|}{2u^2 + 4a - 1}\, du = \ln\left(\frac{4a - 1}{2 + a}\right).$$

Therefore $\Lambda(\lambda_\perp)$ is negative for $a < 1$. This is another example of where the invariant line is an attractor, but because the singular curve passes through the focal point on the invariant line, there is a region where points escape to another attractor, and basins for the three invariant lines become riddled in the presence of symmetry. Again riddling is not due to loss of symmetry but because of the presence of singular lines.

Here, it is easier to believe that each line attracts a set of positive measure for $\frac{1}{4} < a < 1$. As a exceeds one and the three lines lose their stability, almost all iterates appear to execute ergodic behavior on the entire phase space of the system. Further, since there are no other attractors present and there are saddles on each of the invariant lines, again riddling seems to be associated with sets of initial conditions having zero measure that escape to other basins.

This article presented two examples of a riddling bifurcation caused by singularities. While few problems lend themselves to the exact determination of their Lyapunov exponents, here we employ this property to accentuate the unexpected riddling of a basin of attraction in the presence of negative transverse Lyapunov exponents after the bifurcation. For noninvertible maps, other important elements contributing to the riddling include focal points and bow ties. We also conjecture that the new basin has positive measure in the plane. Riddling is a complicated phenomenon and by no means fully understood and clearly deserving of additional study.

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