Reentrance Phenomena in a Bistable Kinetic Model Driven by Correlated Noise

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In this Letter we discuss a bistable model driven by two white noise sources, when the correlation time of the correlations between two sources τ is nonzero. We find that there is a critical value of the correlation time τ_c. For $\tau > \tau_c$, the system undergoes a succession of two phase transitions (namely, the reentrance phenomenon) as the strength of the correlations between two noise sources λ is varied. However, for $\tau < \tau_c$, the system undergoes a (single) phase transition as λ is varied. [S0031-9007(96)01943-6]

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The study of dynamical systems disturbed by noise is recurrent in many contexts of physics and other sciences. Particularly for nonequilibrium systems, where noise plays a crucial role[1,2], the noise-induced transition has been intensively investigated. The effects of colored noise (i.e., fluctuations of intensity D with a correlation time τ) have attracted a great deal of interest in recent years. Different theories have been used to deal with the colored-noise problem, for instance, the conventional small-τ theory [3,4], the functional-calculus theory of Fox [5], the decoupling theory (often called Hanggi ansatz) [6], the unified colored-noise theory [7], and the recent interpolation procedure [8].

Recently, Castro *et al.* [9] presented an analysis of a chemical reaction system, when it is forced by one colored-noise source, by using the interpolation procedure which is an extension of the unified colored-noise theory and allows us to study both $\tau \to 0$ and $\tau \to \infty$ cases. They showed that the system undergoes a purely noise-induced transition from a monostable regime to a bistable one, then to the monostable regime as τ increases, namely, a succession of two phase transitions when the correlation time is monotonically varied. This type of nonequilibrium transitions phenomenon has been called the reentrance phenomenon [9].

The largest amount of work about fluctuations has been referred to the consideration of systems with just one noise source. However, more realistic models of physical systems require considering various noise sources, for example, the laser models [10], the lattice model [11], the structure-formation process in liquid crystals [12], the imperfect pitchfork bifurcation in superfluid turbulence in liquid helium [13], etc. Though various sources are presented simultaneously in some stochastic processes, they are assumed to have different origins and are treated as independent random variables in most of the previous investigations [10–15]. However, in certain situations they may have a common origin and thus may be correlated with each other as well [16–21]. The study of dynamical systems with correlation noise terms has attracted attention in the field of stochastic processes. Some of these investigations were concerned with the steady-state statistical properties of systems [16–20]; others were concerned with the transient problems [21]. Now a question to be raised is if similar peculiarities studied by Castro *et al.* [9] can be found in other models driven by correlated noise.

In this Letter we analyze a general bistable system driven by two white noise sources, when the correlation time of the correlations between the two sources is nonzero, by using the conventional small-τ theory [4]. Our results show that a novel feature corresponding to the reentrancelike phenomenon is indicated by the phase diagram of the system.

A typical case with correlation noise terms is described by the following general stochastic differential equation

$$\dot{x}(t) = f(x) + g_1(x)\xi(t) + g_2(x)\eta(t), \qquad (1)$$

where $\xi(t)$ and $\eta(t)$ are Gaussian white noise sources with zero mean, and

$$\langle \xi(t)\xi(t')\rangle = 2D\,\delta(t - t'), \qquad (2)$$

$$\langle \eta(t)\eta(t')\rangle = 2Q\,\delta(t - t'), \qquad (3)$$

where D and Q are the intensities of the noise. Here we assume

$$\langle \xi(t)\eta(t')\rangle = \langle \eta(t)\xi(t')\rangle = \frac{\lambda\sqrt{QD}}{\tau}\exp[-|t - t'|/\tau] \to 2\lambda\sqrt{QD}\,\delta(t - t'), \qquad \text{as } \tau \to 0, \qquad (4)$$

in which τ is the correlation time and λ is the strength of the correlations between $\xi(t)$ and $\eta(t)$.

A general equation satisfied by the probability of the process (1) with (2)–(4) is given by [4]

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}f(x)P(x,t) - \frac{\partial}{\partial x}g_1(x) \times \langle \xi(t)\delta[x(t) - x]\rangle - \frac{\partial}{\partial x}g_2(x)\langle \eta(t)\delta[x(t) - x]\rangle, \qquad (5)$$

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where $P(x,t) = \langle \delta[x(t) - x] \rangle$; the average (5) can be calculated for Gaussian noise $\eta(t)$ and $\xi(t)$ by the Novikov theorem [22]. The Fokker-Planck equation of the small-τ approximation for (1) is obtained following Ref. [4]:

$$\begin{aligned}\frac{\partial}{\partial t} P(x,t) = & -\frac{\partial}{\partial x} f(x) P(x,t) \\ & + D \frac{\partial}{\partial x} g_1(x) \frac{\partial}{\partial x} g_1(x) P(x,t) \\ & + \lambda \sqrt{QD} \frac{\partial}{\partial x} g_1(x) \frac{\partial}{\partial x} h_2(x) P(x,t) \\ & + Q \frac{\partial}{\partial x} g_2(x) \frac{\partial}{\partial x} g_2(x) P(x,t) \\ & + \lambda \sqrt{QD} \frac{\partial}{\partial x} g_2(x) \frac{\partial}{\partial x} h_1(x) P(x,t), \quad (6)\end{aligned}$$

with

$$h_1(x) = g_1(x)\{1 + \tau g_1(x)[f(x)/g_1(x)]'\}, \quad (7)$$

$$h_2(x) = g_2(x)\{1 + \tau g_2(x)[f(x)/g_2(x)]'\}. \quad (8)$$

Consider now the single bistable kinetic model [14,17–19]

$$\dot{x}(t) = -ax - bx^3 - x\xi(t) + \eta(t) \quad (a < 0, b > 0). \quad (9)$$

Equation (9) is a special case of Eq. (1) with $f(x) = -ax - bx^3$, $g_1(x) = -x$, and $g_2(x) = 1$. The Fokker-Planck equation can be obtained from (6)

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[F(x) + \frac{\partial}{\partial x} G(x)\right] P(x,t), \quad (10)$$

where

$$F(x) = bx^3 - 3b\tau\lambda\sqrt{QD}\,x^2 + (a - D)x + \lambda\sqrt{QD}\,(1 + a\tau), \quad (11)$$

$$G(x) = 5b\tau\lambda\sqrt{QD}\,x^3 + Dx^2 + \lambda\sqrt{QD}\,(a\tau - 2)x + Q. \quad (12)$$

The steady-state distribution of Eq. (10) is

$$P_{\rm st}(x) = N[G(x)]^{-1} \exp\left[-\int^x \frac{dxF(x)}{G(x)}\right]. \quad (13)$$

Although we cannot give the explicit expression of $P_{\rm st}(x)$ here, yet we can give two important results.

(i) The extrema of the steady-state distribution (13) are determined by the following equation of third order:

$$\begin{aligned}[x + 4\tau\lambda\sqrt{QD}]^3 + [(a + D)/b - 48b\tau^2\lambda^2 QD] \\ \times [x + 4\tau\lambda\sqrt{QD}] + 128\tau^3\lambda^3(QD)^{3/2} \\ - \lambda\sqrt{QD}\,(1 + 2a\tau + 4D\tau)/b = 0. \quad (14)\end{aligned}$$

(ii) The equation of the critical parameter a_c at which the transition between the monostable ($a > a_c$) and the bistable ($a < a_c$) distribution occurs:

$$\frac{1}{4}\left[128\tau^3\lambda^3(QD)^{3/2} - \frac{\lambda\sqrt{QD}\,(1 + 2a_c\tau + 4D\tau)}{b}\right]^2 + \frac{1}{27}\left[\frac{a_c + D}{b} - 48b\tau^2\lambda^2 QD\right]^3 = 0. \quad (15)$$

The calculations presented in this Letter are rather standard. However, the conclusions that can be drawn from the above results are interesting.

When the correlation time τ is fixed, we have plotted the curve of the critical value a_c of the steering parameter a as a function of the strength of correlations between two sources λ in Fig. 1(a). Remember that the critical value a_c is negative for the bistable system Eq. (9). The presence of the correlation between two sources causes the critical curve separating the bistable region (above the curve) and monostable region (below the curve) in the phase diagram of the system. The phase diagram makes it apparent that, for some regions of values of a_c and for small values of the strength λ, the system is found in a bistable phase (P_1). After increasing λ beyond some threshold value, the system undergoes a transition to a monostable phase (P_2). However, if we further increase λ, the system goes back to a bistable phase (P_1); that is, the system undergoes a succession of two phase transitions ($P_1 \rightarrow P_2 \rightarrow P_1$) when the strength of correlation between two sources λ is monotonically varied. The horizontal thin line in Fig. 1(a) corresponds to a path of a succession of two phase transitions.

A natural question is whether this reentrancelike phenomenon can also occur in the whole regions of values of correlation time. In Fig. 1(b) we have plotted the curves of a_c as a function of λ for several values of correlation time. We find that there is a critical value of correlation time τ_c [which is determined by Eq. (15)], the reentrance phenomenon exists only for $\tau > \tau_c$ as the strength λ is varied [e.g., the case of $\tau = 0.2$ or 0.3 in Fig. 1(b)]. Otherwise there is only a (single) phase transition as λ is varied [e.g., the case of $\tau = 0.0$ or 0.1 in Fig. 1(b)]. Moreover, if the system is situated under the $\tau = 0.0$ curve (e.g., point *M*), fixing the strength λ, an increasing correlation time τ will only lead to a (single) transition from the monostable phase to a bistable one.

Although the reentrance phenomenon is studied by Castro *et al.* [9] and by us, respectively, yet there is an essential difference. The phenomenon studied by Castro *et al.* originates from the color effects of the colored noise, but the phenomenon studied by us originates from the correlative effects and the color effects of the correlations between two white noise sources. In addition, Castro *et al.* [9] show that the reentrance phenomenon occurs as the correlation time of colored noise varies, while we find that this phenomenon can also occur as the strength of the correlations between two white noise sources varies. Therefore the reentrance phenomenon studied by us is not exactly the same as the one studied by Castro *et al.* [9], but is nevertheless interesting.

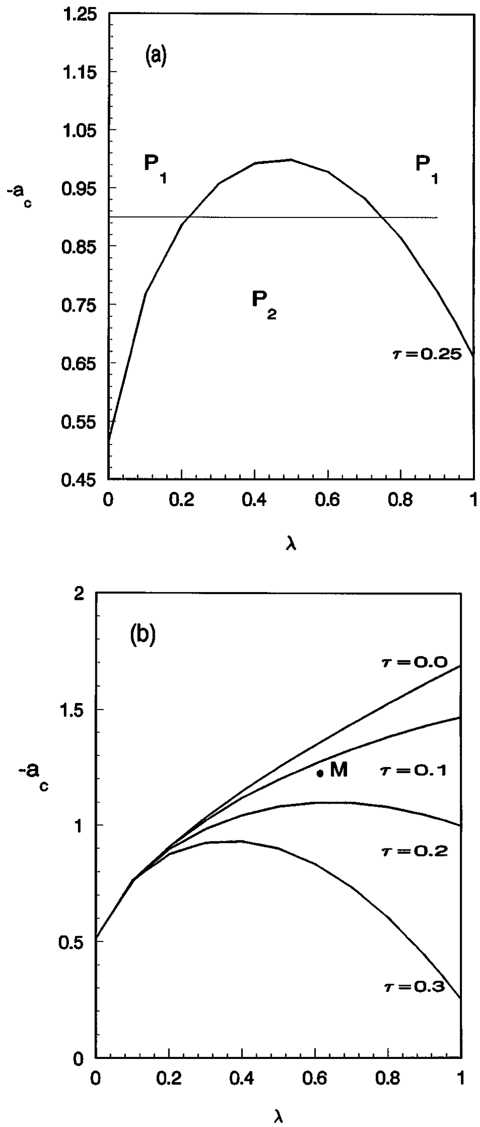

FIG. 1. Phase diagram of the system. The parameter values are $b = 1$, $Q = 0.5$, and $D = 0.5$. (a) P_1 denotes the bistable phase and P_2 denotes the monostable one. The horizontal thin line corresponds to the path of a succession of two phase transitions ($P_1 \rightarrow P_2 \rightarrow P_1$). (b) The phase diagram of the system for several values of the correlation time: $\tau = 0.0$, $\tau = 0.1$, $\tau = 0.2$, and $\tau = 0.3$. If the system is situated at point M, the system is found in a monostable phase (P_2) for $\tau = 0.0$ or $\tau = 0.1$, but in a bistable phase (P_1) for $\tau = 0.2$ or 0.3.

Because the bistable model is of generic interest in physics and other science, and more realistic models of physical systems require considering various noise sources simultaneously as mentioned in our introduction, so that our results probably apply to a broader class. Furthermore, in our case, the dependence of the reentrance phenomenon on the macrovariable a (the steering parameter of the bistable model) and the strength of correlations λ would make the control of the phenomenon easier in experimental search.

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[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry and Biology* (Springer-Verlag, Berlin, 1984).
[2] F. Moss and P.V.E. McClintock, *Noise in Nonlinear Dynamical Systems* (Cambridge University Press, Cambridge, 1989), Vols. I and II.
[3] R.L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. I.
[4] J.M. Sancho, M. San Miguel, S.L. Katz, and J.P. Gunton, Phys. Rev. A **26**, 1589 (1982).
[5] R.F. Fox, Phys. Rev. A **33**, 467 (1986); **34**, 4525 (1986); J. Stat. Phys. **46**, 1145 (1987).
[6] P. Hanggi, T.T. Mrocakowski, F. Moss, and P.V.E. McClintock, Phys. Rev. A **32**, 695 (1985).
[7] P. Jung and P. Hanggi, Phys. Rev. A **35**, 4464 (1987); J. Opt. Soc. Am. B **5**, 979 (1988).
[8] F. Castro, H.S. Wio, and G. Abramson, Phys. Rev. E **52**, 159 (1995).
[9] F. Castro, A.D. Sanchez, and H.S. Wio, Phys. Rev. Lett. **75**, 1691 (1995).
[10] R. Roy, A.W. Yu, and S. Zhu, Phys. Rev. Lett. **55**, 2794 (1985); S. Zhu, A.W. Yu, and R. Roy, Phys. Rev. A **34**, 4333 (1986); R.F. Fox and R. Roy, *ibid.* **35**, 1838 (1987); R.F. Fox, *ibid.* **34**, 3405 (1986); M.R. Young and S. Singh, *ibid.* **38**, 239 (1988); E. Peacock-Lopez, F.J. de la Rubia, B.J. West, and K. Lindenberg, *ibid.* **39**, 4026 (1989).
[11] C. Van den Broeck, J.M.R. Parrondo, J. Armero, and A. Hernandez-Machado, Phys. Rev. E **49**, 2639 (1994).
[12] S. Kai, S. Wakabayashi, and M. Imasaki, Phys. Rev. A **33**, 2612 (1986).
[13] M.F. Schumaker and W. Horsthemke, Phys. Rev. A **36**, 354 (1987); D. Griswold and J.T. Tough, *ibid.* **36**, 1360 (1987).
[14] J.J. Brey, J.M. Casado, and M. Morillo, Z. Phys. B **66**, 263 (1987).
[15] N.G. Stocks, R. Mannella, and P.V.E. McClintock, Phys. Rev. A **42**, 3356 (1990); C.R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994); F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, and S. Santucci, Phys. Lett. A **201**, 275 (1995).
[16] R.F. Fox, Phys. Lett. **94A**, 281 (1983); I.I. Fedchenia, J. Stat. Phys. **50**, 1043 (1988); **52**, 1005 (1988); Shiqun Zhu, Phys. Rev. A **47**, 2405 (1993); Da-jin Wu, Li Cao, and Sheng-zhi Ke, Phys. Rev. E **50**, 2496 (1994); **52**, 3228 (1995); Jin Wang and Shiqun Zhu, Phys. Lett. A **207**, 47 (1995); Jin Wang, Shiqun Zhu, and Jianping Yin, Phys. Rev. A **51**, 5035 (1995); Jin-hui Li and Zu-qia Huang, Phys. Rev. E **53**, 3315 (1996); Ya Jia, Commun. Theor. Phys. **23**, 45 (1995).
[17] A. Fulinski and T. Telejko, Phys. Lett. A **152**, 11 (1991).
[18] Li Cao and Da-jin Wu, Phys. Lett. A **185**, 59 (1994).
[19] Xue-qin Wei, Li Cao, and Da-jin Wu, Phys. Lett. A **207**, 338 (1995).
[20] Ya Jia and Jia-rong Li, Phys. Rev. E **53**, 5786 (1996).

[21] Ya Jia, Li Cao, and Da-jin Wu, Phys. Rev. A **51**, 3196 (1995); Ya Jia and Jia-rong Li, Phys. Rev. E **53**, 5764 (1996).
[22] E. A. Novikov, Sov. Phys. JETP **20**, 1290 (1965).