Fractal Model for Coarse-Grained Nonlinear Partial Differential Equations

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Spatially coarse-grained (or effective) versions of nonlinear partial differential equations must be closed with a model for the unresolved small scales. For systems that are known to display fractal scaling, we propose a model based on synthetically generating a scale-invariant field at small scales using fractal interpolation, and then analytically evaluating its effects on the large, resolved scales. The procedure is illustrated for the forced Burgers equation, solved numerically on a coarse grid. Detailed comparisons with direct simulation of the full Burgers equation and with an effective viscosity model are presented. [S0031-9007(96)02215-6]

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Many physical processes obey reliably known dynamical equations whose solutions involve a wide range of active length and time scales. This often makes analytical or direct numerical solution procedures impossible or, at best, impractical. Solving more economical "coarse-grained" versions of the equations can yield results that convey much of the important and practically relevant large-scale features of the full solution. However, unless the dynamical equations are linear, the success of such a program depends largely upon the realism with which the effects of unresolved phenomena are represented. For phenomena without "scale separation"—hydrodynamic turbulence [1] and domain growth in random media [2] are familiar examples - such modeling is a challenge.

Within the general subject area of "nonlinear physics" dealing with complex phenomena lacking scale separation, abundant empirical observations have been made about scale invariance (or fractality) as some parameter approaches a critical value [3]. However, the crucial task of translating such observations into a workable mathematical framework that utilizes the dynamical equations (employing fractals in a *predictive* fashion) has proven to be quite difficult in most instances. In this Letter, we describe an approach that uses fractals to close coarse-grained, nonlinear partial differential equations (PDEs) that govern processes which display scale invariance (fractality) at small length scales. We begin by describing the basic idea in a general setting, but soon specialize it to a simple one-dimensional case, namely, the forced Burgers equation.

We begin with a field equation of the form

$$\frac{\partial}{\partial t}\mathbf{a}(\mathbf{x},t) = \mathcal{L}[\mathbf{a}] + \mathcal{N}[\mathbf{a}], \tag{1}$$

which, when endowed with appropriate initial and boundary conditions, governs $\mathbf{a}(\mathbf{x},t)$, a time-dependent vector field in some spatial domain of characteristic length $O(L)$. $\mathcal{N}$ and $\mathcal{L}$ are nonlinear and linear operators, respectively. Let us assume that the solution of this equation has complex spatial structure down to some small cutoff scale $\eta \ll L$. Full numerical solution may be prohibitive, requiring $O(L/\eta)$ elements in each spatial direction. Convolution of $\mathbf{a}(\mathbf{x},t)$ with $G_\Delta(\mathbf{x})$, a homogeneous spatial filter with characteristic scale Δ ($\eta \ll \Delta \ll L$), yields a coarse-grained variable $\tilde{\mathbf{a}} \equiv G_\Delta \mathbf{a}$. Once an effective equation for $\tilde{\mathbf{a}}$ is found, it is amenable to a now less demanding numerical integration on a mesh with resolution of order Δ. The effective equation is derived by convolving the original one with the filter G_Δ, which yields

$$\frac{\partial}{\partial t}\tilde{\mathbf{a}}(\mathbf{x},t) = \mathcal{L}[\tilde{\mathbf{a}}] + \mathcal{N}[\tilde{\mathbf{a}}] + S[\cdot], \tag{2}$$

where $S[\cdot] \equiv \widetilde{\mathcal{N}[\mathbf{a}]} - \mathcal{N}[\tilde{\mathbf{a}}]$. This situation is encountered, for example, when coarse graining hydrodynamic equations to eliminate part of the spatial complexity of turbulence (large-eddy simulation) [4].

The basic difficulty resides in expressing S as a function of $\tilde{\mathbf{a}}$. For this purpose, physical and/or statistical information about the small-scale field $\mathbf{a}' \equiv \mathbf{a} - \tilde{\mathbf{a}}$ is required. For instance, in kinetic theory one uses the assumption of scale separation between molecular and macroscopic variables to derive closures (e.g., derive Navier-Stokes from Boltzmann equation [5]), with *a posteriori* verification that the equations derived do not lead to violation of the assumed scale separation (e.g., do not develop singularities). Our approach is based on using an assumption of scale invariance for the (restricted) set of phenomena which display self-similar (fractal) behavior. We will also verify the self-consistency of the resulting effective equations. The essence of the proposed method is to construct the unknown small-scale portion of the field $\mathbf{a}(\mathbf{x},t)$ by "extrapolating" features of the coarse-grained field $\tilde{\mathbf{a}}(\mathbf{x},t)$ to smaller and smaller scales. A mathematical tool which allows one to generate such a synthetic small-scale field is the so-called "fractal interpolation technique" [6]. It is based on a mapping $\mathbf{W}[\cdot]$ which transforms the features of $\tilde{\mathbf{a}}$ at scales even coarser than Δ onto the complete signal $\tilde{\mathbf{a}}$. More concretely, if $\overline{\tilde{\mathbf{a}}}$ represents the field $\tilde{\mathbf{a}}$ at some resolution $\Delta' > \Delta$, coarser than the basic resolution Δ, the mapping generates $\tilde{\mathbf{a}} = \mathbf{W}[\overline{\tilde{\mathbf{a}}}]$. In order to generate a "synthetic" small-scale field,

this mapping is simply iterated many more times; i.e., we generate a fractal field $\mathbf{a}_f(\mathbf{x},t) = \lim_{n\to\infty}\mathbf{W}^{(n)}[\tilde{\mathbf{a}}] \equiv \mathbf{W}[\mathbf{W}[\mathbf{W}[\cdots\mathbf{W}[\tilde{\mathbf{a}}]\cdots]]]$ (other means of generating synthetic turbulence have been proposed [7], but fractal interpolation will be shown to be particularly well suited for our purposes). Once we have generated the full field, the unknown term $\widetilde{\mathcal{N}[\mathbf{a}]} - \mathcal{N}[\tilde{\mathbf{a}}]$ is evaluated according to its definition. One important element in our proposed approach is that we shall be able to evaluate this term analytically from the parameters of the mapping $\mathbf{W}$ (which depend on $\tilde{\mathbf{a}}$ and the fractal dimension), without the need to explicitly construct the small-scale field. (Notice that such an explicit or numerical construction would require a computational mesh at scales much smaller than Δ, eliminating the usefulness of coarse graining in the first place.) Once the unknown term $\widetilde{\mathcal{N}[\mathbf{a}]} - \mathcal{N}[\tilde{\mathbf{a}}]$ is expressed in terms of $\tilde{\mathbf{a}}$, it is replaced into the dynamical coarse-grained equation, which may then be solved.

While the method is, in principle, quite general, we now focus on a simple one-dimensional hydrodynamic problem, the randomly forced, viscous Burgers equation with periodic boundary conditions in the interval $[0,1]$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}uu\right) = \nu_0\frac{\partial^2 u}{\partial x^2} + f(x,t)\,. \qquad (3)$$

This equation is also the prototype of a class of equations that governs a range of phenomena, from stochastic growth of interfaces to directed polymers [2]. Recent work [8] shows that if the forcing has zero mean and a spectrum of the form $\langle\hat{f}(k,\omega)\hat{f}(k',\omega')\rangle = \mathcal{A}k^{-1}\delta(k+k')\delta(\omega+\omega')$ (i.e., is white in time but has spatial correlations), Kolmogorov-like scaling of the resulting velocity is obtained ($\langle|\hat{u}(k)|^2\rangle \sim k^{-5/3}$) for $k \ll \eta^{-1}$, where η is a viscous cutoff scale. One also finds that the second-order structure function follows a scaling $\langle[u(x+r)-u(x)]^2\rangle \sim r^{2/3}$, which suggests that the fractal dimension of the signal $u(x)$ is $D = 5/3$ (not unlike the case of 3D turbulence [9]). Therefore, this is an application where we know that asymptotically ($\nu_0 \to 0$) the solution at small scales is scale invariant. Our method explicitly uses this information to model the small-scale features of the system.

The effective equation for the coarse-grained 1D velocity $\tilde{u}(x,t)$ is found by convolving Eq. (3) with a top-hat filter of size Δ [$G_\Delta(x) = \Delta^{-1}$ if $|x| < \Delta/2$, zero otherwise],

$$\frac{\partial\tilde{u}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}\tilde{u}\tilde{u}\right) = \tilde{f}(x,t) - \frac{1}{2}\frac{\partial\tau}{\partial x} + \nu_0\frac{\partial^2\tilde{u}}{\partial^2 x}\,, \quad (4)$$

where $\tau = \widetilde{u^2} - \tilde{u}^2$. While one can certainly obtain numerical solutions to this equation by using an effective viscosity model, e.g., $\tau = -\nu_{\text{eff}}\tilde{u}_x$, such a model can only be justified for cases with significant scale separation and has little basis in the present application.

Instead, let us employ the basic idea underlying fractal interpolation [6], which is to replicate on smaller and smaller scales features of $\tilde{u}$ present at a coarse scale. Let us discretize the unit interval in Δ^{-1} segments, each centered at position x_i. For simplicity, let us define a local coordinate $\xi = (x - x_{i-1})/2\Delta$ which goes from $\xi = 0$ to $\xi = 1$ between points x_{i-1} to x_{i+1}. The piecewise linear mapping to generate the locally fractal field in the interval $[x_{i-1}, x_{i+1}]$ is given by

$$W_i[u] = \begin{cases} d_{i,1}u(2\xi) + a_{i,1}\xi + b_{i,1}, & \text{if } \xi\in(0,\frac{1}{2})\,, \\ d_{i,2}u(2\xi-1) + a_{i,2}\xi + b_{i,2}, & \text{if } \xi\in(\frac{1}{2},1)\,. \end{cases} \qquad (5)$$

The parameters $a_{i,j}$ and $b_{i,j}$ are related to the known velocities according to $a_{i,1} = 2[\tilde{u}_{i-1} - \tilde{u}_i - d_{i,1}(\tilde{u}_{i+1} - \tilde{u}_{i-1})]$, $a_{i,2} = 2[\tilde{u}_{i+1} - \tilde{u}_i - d_{i,2}(\tilde{u}_{i+1} - \tilde{u}_{i-1})]$, $b_{i,1} = \tilde{u}_{i-1}(1 - d_{i,1})$, and $b_{i,2} = \tilde{u}_i - d_{i,2}\tilde{u}_{i-1}$. The parameters $d_{i,1}$ and $d_{i,2}$ determine the vertical stretching of the left and right segments at each iteration, and they must obey $|d_i| < 1$ for the mapping to be contracting (i.e., for a fixed point to exist). Repeated application of this mapping generates a fractal function as the fixed point of the map, $u_f(\xi) = W_i[u_f(\xi)]$, which passes through all the points $(x_i, \tilde{u}_i)$, but has fluctuations down to much smaller scales. The d_i's are related to D, the fractal dimension of $u_f(x)$, by $|d_1| + |d_2| = 2^{D-1}$. Figure 1 illustrates the first two steps of the construction process, as well as the 10th iteration of the map.

Interestingly, it is now possible to evaluate, by recurrence, a variety of integrals of the fractal function. For instance, solving for $\int_0^1 u_f(\xi)\,d\xi$ yields

$$\int_0^1 u_f\,d\xi = \frac{\tilde{u}_{i-1} + 2\tilde{u}_i + \tilde{u}_{i+1}}{2(2 - d_{i,1} - d_{i,2})} - \frac{(d_{i,1} + d_{i,2})(\tilde{u}_{i+1} + \tilde{u}_{i-1})}{2(2 - d_{i,1} - d_{i,2})}\,. \qquad (6)$$

We set $d_{i,1} = -d_{i,2}$, because then Eq. (6) shows that the integral (i.e., the synthetic velocity box filtered at scale 2Δ) reduces to the integral of the piecewise linear

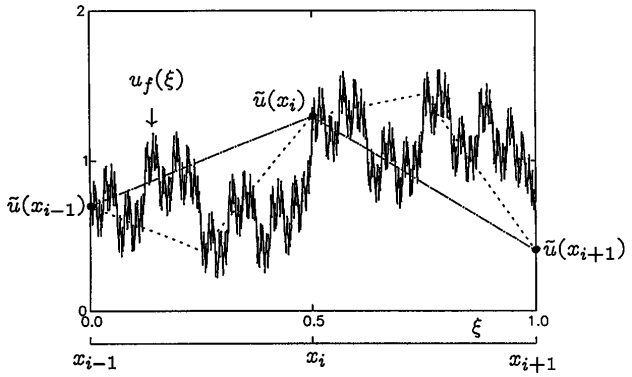

FIG. 1. Different states during the construction of the fractal function $u_f(\xi)$, which interpolates between $\tilde{u}_{i-1}$, $\tilde{u}_i$, and $\tilde{u}_{i+1}$. If $u_0(\xi) = (\tilde{u}_{i+1} - \tilde{u}_{i-1})\xi + \tilde{u}_{i-1}$, then the dash-dotted line is $W_i[u_0(\xi)]$, with W_i defined as in Eq. (5). The dotted line is $W_i[W_i[u_0(\xi)]]$ and the continuous curve is $u_f(\xi) \simeq W_i^{10}[u_0(\xi)]$. The dots mark the points to be interpolated. In this example $D = \frac{5}{3}$ ($d_{i,1} = -2^{-1/3}$, $d_{i,2} = 2^{-1/3}$).

interpolation (i.e., the velocity $\tilde{u}$ filtered at scale 2Δ). Moreover, this recursive technique can be used to evaluate any (local) moment of fractally interpolated functions over diadic subintervals [6], such as $\int_0^{1/2}(\xi)^p[u_f(\xi)]^q\,d\xi$ and $\int_{1/2}^1(\xi)^p[u_f(\xi)]^q\,d\xi$. This is quite useful because the unknown subgrid stress τ produced by the synthetic field around x_i can be written as $\tau_i = \int_{1/4}^{3/4}[u_f(\xi)]^2\,d\xi - [\int_{1/4}^{3/4}u_f(\xi)\,d\xi]^2$. After some algebra, one obtains

$$\tau_i[\tilde{u}, d_i] = \frac{1}{12}(\delta_i\tilde{u})^2 + \frac{d_i(8-3d_i^2)}{48}\delta_i^2\tilde{u}\delta_i\tilde{u} + \frac{1+15d_i^2-24d_i^4+12d_i^6}{192(1-d_i^2)}(\delta_i^2\tilde{u})^2. \quad (7)$$

Above, $\delta_i\tilde{u} = (\tilde{u}_{i+1} - \tilde{u}_{i-1})/2$, $\delta_i^2\tilde{u} = \tilde{u}_{i+1} - 2\tilde{u}_i + \tilde{u}_{i-1}$, and $d_i = \pm 2^{D-2}$, where the sign will be chosen at random with equal probabilities. Still, the fractal dimension D is needed in order to evaluate τ_i. Instead of prescribing D (or d_i) *a priori,* a dynamical equation for τ can be combined with Eq. (7) to self-consistently solve for τ and d. From Burgers equation, we find

$$\frac{\partial\tau}{\partial t} + \tilde{u}\frac{\partial\tau}{\partial x} = -\varepsilon - 2\frac{\partial\tilde{u}}{\partial x}\tau - 2\frac{\partial A}{\partial x} + 2(\widetilde{fu} - \tilde{u}\tilde{f}). \quad (8)$$

The flux $A = (\widetilde{u^3} - \tilde{u}^3)/3 - \tilde{u}\tau$ and small-scale forcing $\widetilde{fu} - \tilde{f}\tilde{u}$ are computed in a straightforward fashion with the technique outlined above [10]. $\varepsilon = 2\nu_0[(\widetilde{\partial u/\partial x})^2 - (\partial\tilde{u}/\partial x)^2]$ is the dissipation due to molecular viscosity and requires a more careful analysis since the fractal signal is not differentiable if $d > 1/2$ (or $1 < D < 2$, which is, of course, the interesting case). However, if we stop the iteration of the map at the nth step, i.e., we introduce a cutoff scale at $\eta \sim 2^{-n}\Delta$, we can evaluate integrals of $(du_f/dx)^2$ and obtain

$$\varepsilon_i = 2\nu_0\left[\frac{1}{4} + 4d_i^4\frac{(4d_i^2)^{n+1}-1}{4d_i^2-1}\right](\delta_i^2\tilde{u})^2. \quad (9)$$

For forced Burgers equation with Kolmogorov scaling, the usual dimensional argument yields $\eta = C(\nu_0^3/\varepsilon)^{1/4}$, with $C = O(1)$. Besides the geometric rules of our fractal interpolation scheme, this step is the only physical modeling required. Combining this (with $C = 1$) and Eq. (9) we obtain an equation for the dissipation solely in terms of d and ν_0. As an aside, it can be easily shown that $\lim_{\nu_0\to 0}\varepsilon(d,\nu_0) = O(\nu_0^\beta)$ with $\beta = 1$ if $d < 1/2$ or $\beta = -\frac{3}{2}\log_2(2^{1/3}d)$ if $d > 1/2$ which implies that $d = 2^{-1/3}$ (or $D = \frac{5}{3}$) is the cross-over scaling at which a fractal signal dissipates energy even in the inviscid limit [11]. Different dissipation mechanisms (such as hyperviscosity) would lead to different expressions for ε in Eq. (9) and to a different estimate for η. This, in turn, may affect the value of d.

In summary, we have a system of four equations (4), (7), (8), and (9) in the four unknowns $(\tilde{u}, \tau, d, \varepsilon)$, and are now in a position to explore numerical solutions of these equations. We use a pseudospectral code on the unit interval to solve for $\tilde{u}(x,t)$ and $\tau(x,t)$, using 2^7 modes (i.e., $\Delta = \frac{1}{256}$). This scale should be small enough for a scaling range to develop in the resolved field. The time advancement is based on the Adam-Bashfort 2 scheme. The forcing uses $\mathcal{A} = \sqrt{2}\times 10^{-3}$, the time step is $\Delta t = 5.0\times 10^{-5}$ [resulting in a Courant-Friedricks-Lewey (CFL) number much less than unity), and the viscosity is $\nu_0 = 1.0\times 10^{-5}$. This value implies an expected cutoff scale for the real Burgers equation of about $\eta \sim \frac{1}{4096}$, much smaller than our coarse resolution. The random forcing is generated in Fourier space by superposing Fourier modes with prescribed amplitudes equal to $A/\sqrt{\Delta t}$ and phases θ, chosen randomly at each time step with a uniform weight in $[0, 2\pi]$. The model expressions are evaluated in physical space, but transformed to Fourier space for differentiation. The nonlinear terms are dealiased by the 2 rule (zero padding), and the minimum attainable value of τ (realizability) is enforced numerically [12]. We found that Eq. (7) always had a unique solution for d_i as a function of τ_i with $d_i \in [0, 1]$. Several simulations with different initial conditions have been performed. For example, if we set $\tilde{u}(x, t=0)$ to be a random signal with a Kolmogorov spectrum, after an initial decay the global kinetic energy $e_g = \frac{1}{2}\int_0^1\tilde{u}^2\,dx$ reaches a plateau where forcing and dissipation are in approximate equilibrium. The squares in Fig. 2 show the time evolution of e_g, averaged over three different realizations of the forcing.

In order to illustrate the impact of the fractal subgrid model, we also perform a simulation with $\tau = 0$. As expected, the energy injected cannot be dissipated, and the simulation blows up (dash-dotted line). For purposes of comparison, we also perform a direct simulation [of Eq. (3)] with no coarse graining or modeling, using 2^{13}

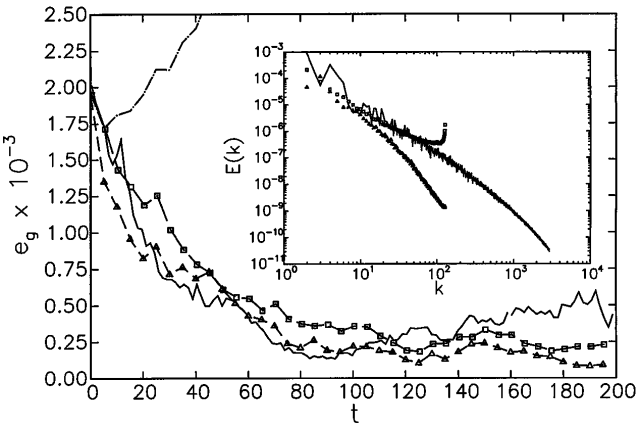

FIG. 2. Kinetic energy e_g versus time for fully resolved equation (solid curve), fractal model (squares), effective viscosity model (triangles), and coarse-grained solution without modeling (dash-dotted line). The inset shows the energy spectrum (same symbols) averaged in time between $60 < t < 200$ and over different realizations of the coarse-grained simulation, achieving improved convergence at low k.

modes and the same random forcing for the common, large-scale, modes. For the present value of ν_0 the full simulation requires roughly 10 times the memory as compared with the model and takes 4 times longer per time step. As shown in Fig. 2, solid line, the evolution of the global energy of the direct simulation and that of the coarse-grained simulations are quite close. Finally, we have performed a simulation where τ is replaced by an effective viscosity term (which runs about 5 times faster than the full fractal model), with the eddy viscosity $\nu_{\text{eff}} = 5.0 \times 10^{-3}$ adjusted so as to dissipate energy at the rate predicted by the direct numerical simulation (DNS) (Fig. 2, triangles). We have also compared energy spectra (inset, Fig. 2). The fractal model displays good agreement with the DNS up to a wave number $k \sim \pi/2\Delta$, about half of the cutoff wave number. At the smallest octave of resolved scales, the spectrum of the coarse-grained simulation tends to fall somewhat above that of the direct simulation, an indication that the fractal model does not reproduce all the features of the small scales perfectly. However, this has to be contrasted with the much stronger departure of the eddy viscosity simulation which excessively damps the solution. While the spectrum can be made to approach the DNS using smaller values for the eddy viscosity, the energy decay rate becomes significantly smaller and no longer matches that of the DNS.

The fractal dimension calculated shows that the portion of the signal where $d > 2^{-1/3}$, that is to say, where most of the dissipation takes place, covers in average only 25% of the space, with the computed η reaching down to 1.25×10^{-4}. This is consistent with the fact that for the present level of forcing we have found that $\Delta_{\text{DNS}} \sim 1.25 \times 10^{-4}$ is the largest mesh size that can be used to solve Eq. (3) without modeling. Finally, in Fig. 3 we show a segment of the coarse-grained solution $\tilde{u}(x,t)$, represented by the symbols (values at grid points). The solid line is an explicit construction of the synthetic small-scale signal using the fractal interpolation technique (iterating the mapping W_i for n times, with n and d set to the locally computed values). As can be seen, values of n and ε fluctuate greatly from one location to another, indicative of a high level of intermittency.

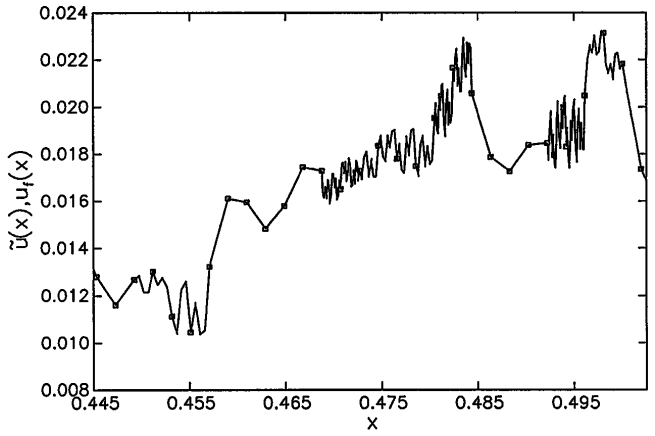

FIG. 3. Sample of large-scale field (squares) as computed from coarse-grained Burgers equation using the fractal model, and explicitly constructed small-scale field (solid curve) using d and n as obtained from the simulation.

We stress again that during the simulation, we did not have to explicitly construct such a small-scale signal, because the fractal interpolation technique allows us to analytically evaluate their effect on the coarse scales directly. As $\tilde{u}$ evolves in time, the synthetic small scales are "slaved" to the resolved ones, and evolve accordingly. The simulation proceeds at the coarse resolution, but the simulated dynamics of $\tilde{u}$ are as if the signal contained all the small-scale fluctuations shown in the figure.

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[1] A. Monin and A. Yaglom, *Statistical Fluid Mechanics* MIT Press, Cambridge, MA, 1971).

[2] A-L Barabasi, G. T. Dewey, and H. E. Stanley, J. Amer. Chem. Soc. **117**, 12 899 (1995); T. Halpin-Healy and Yi-Cheng Zhang, Phys. Rep. **254**, 215–414 (1995); M. Mézard, J. P. Bouchad, and G. Parisi, Phys. Rev. E **52**, 3656–3674 (1995).

[3] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982); J. Feder, *Fractals* (Plenum Press, New York, 1988).

[4] A. Leonard, Adv. Geophys. **18**, 237 (1974); M. Lesieur and O. Metais, Annu. Rev. Fluid Mech. **28**, 45–82 (1996).

[5] C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer-Verlag, Berlin, 1988).

[6] M. F. Barnsley, Constr. Approx. **2**, 303–329 (1986).

[7] A. Juneja, D. P. Lathrop, K. R. Sreenivasan, and G. Stolovitzky, Phys. Rev. E **49**, 5179–5194 (1994); R. Benzi, L. Biferale, A. Crisanti, G. Paladin, M. Vergassola, and A. Vulpiani, Physica (Amsterdam) **65D**, 352–358 (1993).

[8] A. Chekhlov and V. Yakhot, Phys. Rev. E **51**, R1–R4 (1995).

[9] A. Scotti, C. Meneveau, and S. G. Saddoughi, Phys. Rev. E **51**, 5594–5608 (1995).

[10] $A_i[\tilde{u}, d_i] = \delta_i^2\tilde{u}[4(2-d_i^2)(4-2d_i^2+3d_i^4)(\delta_i\tilde{u})^2 + 8d_i \times (8-14d_i^2+13d_i^4-6d_i^6)\delta_i\tilde{u}\delta_i^2\tilde{u} + d_i^2(28-72d_i^2+69d_i^4-36d_i^6)(\delta_i^2\tilde{u})^2]/[3072(2-d_i^2)]$; $\tilde{f}u - \tilde{f}\tilde{u} = \{16(1-d_{i,f}d_i)\delta_i\tilde{f}\delta_i\tilde{u} + d_{i,f}(16-6d_{i,f}^2)(1-d_{i,f}d_i)\delta_i^2\tilde{f}\delta_i\tilde{u} + d_i(16-6d_i^2)(1-d_{i,f}d_i)\delta_i^2\tilde{u}\delta_i\tilde{f} + [(1-d_{i,f}d_i) + d_{i,f}d_i(16+6(d_{i,f}+d_i)^2-12d_{i,f}^2d_i^2)]\delta_i^2\tilde{f}\delta_i^2\tilde{u}\}/[192(1-d_{i,f}d_i)]$, with $d_{i,f} = \pm 1$ with the sign chosen at random.

[11] G. L. Eyink, Physica (Amsterdam) **78D**, 222 (1994).

[12] For numerical reasons, τ is constrained to remain positive by adding a term which is nonzero only when $0 < \tau < \tau^0$. τ^0 is the "trivial" value of the stress when $d = 0$, i.e., $\tau^0 = (\delta_i\tilde{u})^2/12 + (\delta_i^2\tilde{u})^2/192$. The term added is τ^0/τ when $\tau < \tau_0$, zero otherwise. Typically, the condition $\tau < \tau^0$ was observed to occur only 5% of the time, and just on only 3% of the grid points.