Faraday's Instability for Viscous Fluids

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We derive an exact equation which is nonlocal in time for the linear evolution of the surface of a viscous fluid, and show that this equation becomes local and of second order in an interesting limit. We use our local equation to study Faraday's instability in a strongly dissipative regime and find a new scenario which is the analog of the Rayleigh-Taylor instability. Analytic and numerical calculations are presented for the threshold of the forcing and for the most unstable mode with impressive agreement with experiments and numerical work on the exact Navier-Stokes equations. [S0031-9007(96)02234-X]

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If a horizontal fluid layer is vertically oscillated one has Faraday's instability which is well understood for ideal fluids [1]. The interest in viscous fluids is recent [2–8]. The order parameter is the amplitude of the free surface. When dissipation is considered the equation for the amplitude is nonlical in time and it would be a great advantage to have a local and simple equation in order to explore new phenomena. For weak viscosity a phenomenological approach leads to a Mathieu equation with damping which is in good agreement with experiments [2]. This equation is easily obtianed in our formulation. For high vicosity we show here that a simple local equation also exists and is again a Mathieu equation very different from the previous one. This new equation allows one to understand how the instability arises when the system is strongly dissipative: We find that for an acceleration of the plate in the form of a cosinus the mechanism producing the instability looks quite different from the mechanism of parametric resonance, and we interpret it as the analog of the Rayleigh-Taylor instability. Our equation also predicts new phenomena which can be experimentally verified.

We consider a plate with a fluid layer of height h and velocity $\vec{v}$. The vertical axis is $\hat{z}$, $z = 0$ corresponds to the surface of the fluid at rest in contact with the atmosphere, and $z = -h$ is the position of the plate. Let $a(t)$ be the acceleration of the plate, then in the reference frame where the plate is at rest we have for an incompressible fluid ($\nabla \cdot \vec{v} = 0$) the equation [$\vec{x} = (x, y)$ are the horizontal coordinates]

$$\partial_t \vec{v}(\vec{x}, z, t) + [\vec{v}(\vec{x}, z, t) \cdot \nabla]\vec{v}(\vec{x}, z, t) = -\frac{1}{\rho} \nabla p(\vec{x}, z, t) - g_e(t)\hat{z} + \nu \nabla^2 \vec{v}(\vec{x}, z, t), \tag{1}$$

where p is the pressure, ρ the mass density, ν the kinematic viscosity, and $g_e(t)$ is the effective acceleration [gravity g plus the acceleration of the plate $a(t)$]. The basic configuration is the rest state $\vec{v} = 0$ with flat interface and time dependent pressure $p_s = p_0 - \rho g_e(t) z$ ($p_0 =$ atmospheric pressure). Perturbations of the basic configuration, $p = p_s + \rho \pi$, obey linearly

$$(\partial_t - \nu \nabla^2)\vec{v}(\vec{x}, z, t) = -\nabla \pi(\vec{x}, z, t). \tag{2}$$

If ξ denotes the vertical displacement of the upper surface, the linearized boundary conditions (BC), there are [4,8]

$$\begin{aligned} v_z(\vec{x}, z, t)|_{z=0} &= \partial_t \xi(\vec{x}, t), \\ [\pi(\vec{x}, z, t) - 2\nu \partial_z v_z(\vec{x}, z, t)]|_{z=0} &= \left[g_e(t) - \frac{\tau}{\rho} \nabla_\perp^2 \right] \xi(\vec{x}, t), \\ [\partial_x v_z(\vec{x}, z, t) + \partial_z v_x(\vec{x}, z, t)]|_{z=0} &= [\partial_y v_z(\vec{x}, z, t) + \partial_z v_y(\vec{x}, z, t)]|_{z=0} = 0, \end{aligned} \tag{3}$$

where τ is the surface tension and $\nabla_\perp = (\partial_x, \partial_y)$ the horizontal gradient. The first equation (3) is the kinematic condition, and the rest are conditions on the stress tensor $T_{jk} = -p \delta_{jk} + \rho \nu (\partial_j v_k + \partial_k v_j)$. The no slip BC on the plate is $\vec{v}(\vec{x}, z, t)|_{z=-h} = 0$. We shall ignore lateral BC since, for a highly viscous fluid and large aspect ratio, they have no influence [3,5].

Our problem is to solve (2) with its BC. We take $(-\nabla \pi)$ as a source term and write $\vec{v} = \vec{u} + \vec{v}_{\text{par}}$, with $\vec{v}_{\text{par}}$ any particular solution of (2) which we take in gradient form $\vec{v}_{\text{par}} = -\vec{\nabla} \phi$ with $\nabla^2 \phi = 0$, and where $\vec{u}$, which we call the diffusive component of the velocity, is the general solution of the homogeneous diffusion equation (2). Physically, since $\vec{v}_{\text{par}}$ satisfies the ideal fluid equations, we have in $\vec{u}$ the boundary effects [8]. In (3) v_z is privileged and a closed problem with only this component of the velocity is obtained [9] (notice that

$\partial_t \phi = \pi$ is just the Bernoulli equation),

$$\nabla^2 \phi(\vec{x}, z, t) = 0, \quad \partial_t \phi(\vec{x}, z, t) = \pi(\vec{x}, z, t),$$
$$(\partial_t - \nu \nabla^2) u_z(\vec{x}, z, t) = 0, \tag{4}$$

$$[u_z(\vec{x}, z, t)|_{z=0} = \partial_t \xi(\vec{x}, t),$$
$$[\pi(\vec{x}, z, t) - 2\nu \partial_z u_z(\vec{x}, z, t)|_{z=0} = [g_e(t) - (\tau/\rho)\nabla_\perp^2]\xi(\vec{x}, t),$$
$$u_z(\vec{x}, z, t)|_{z=0} = 2\nu \nabla_\perp^2 \xi_k(t),$$
$$\partial_z [u_z(\vec{x}, z, t)|_{z=-h} = [u_z(\vec{x}, z, t)|_{z=-h} = 0. \tag{5}$$

Making a horizontal Fourier transform we have new variables ϕ_k, u_{zk}, etc., and solving $\nabla^2 \phi = 0$ as $\phi_k(z, t) = A_k(t) \cosh k(z + h) + B_k(t) \sinh k(z + h)$, we determine (A_k, B_k) in terms of ξ_k and u_{zk} with the first and the last BC (5). The second BC (5) gives a constraint equation

$$(\partial_t + 2\nu k^2)^2 \xi_k(t) + \omega_k^2(t) \xi_k(t) + \frac{(\partial_t + 2\nu k^2)}{\cosh kh} u_{zk}(z, t)|_{z=-h} + 2\nu k \tanh kh \, \partial_z u_{zk}(z, t)|_{z=0} = 0, \tag{6}$$

where $\omega_k^2(t) \equiv k \tanh kh [g_e(t) + \tau k^2/\rho]$ is the usual frequency of surface waves for constant $g_e(t) = g$. The interpretation of this exact equation is simple: The last two terms represent the effect of the boundaries, and if one eliminates these terms this means that we only take into account dissipation in the region where the velocity is of potential form. The equation without these two terms is the phenomenological approximation of Ref. [2] (the conditions of validity are discussed there), where the amplitude is damped with the well-known factor $e^{-2\nu k^2 t}$ [8].

Equation (6) is not a closed equation for ξ_k, but from (4) and (5) we can determine u_{zk} as a nonlocal functional of $\{\xi_k(\cdot)\}$ solving the problem

$$[\partial_t - \nu(\partial_z^2 - k^2)] u_{zk}(z, t) = 0, \tag{7}$$

$$u_{zk}(z, t)|_{z=0} = -2\nu k^2 \xi_k(t),$$
$$[\sinh kh \, \partial_z u_{zk}(z, t) + k \cosh kh \, u_{zk}(z, t)]|_{z=-h} = -k(\partial_t + 2\nu k^2) \xi_k(t). \tag{8}$$

An exact nonlocal equation for ξ_k is obtained replacing the solution of (7) with BC (8) in (6). Using Laplace transform, we obtain

$$u_{zk}(z, t) = \int_{t_0}^{\infty} dt' \xi_k(t') K(t - t', z) + \int_{-h}^{0} dz' u_{zk}(z', t_0) G(t - t_0, z', z), \tag{9}$$

where $u_{zk}(z, t_0)$ is an initial condition, and the kernels $K(t, z)$ and $G(t, z', z)$ vanish for $t < 0$ (causality) and for $t \to \infty$ due to the finite duration of memory effects. The nonlocality of (6) as an equation for ξ_k is a consequence of (9), and it can be understood by introducing the characteristic time $\tau_D = \ell^2/\nu$, where ℓ is the penetration length of the fluid motion (for example, if $kh \gg 1$ we have $\ell \approx k^{-1}$, and if $kh \ll 1$ then $\ell \approx h$). We identify τ_D as the characteristic time of the memory effects since it represents the time taken by a perturbation to spread in the fluid. If $\tau_D \ll \Omega^{-1}$, where Ω is a characteristic frequency of the system, we can expect (6) to become local in time. Replacing u_{zk} in (6) we obtain [$\mathcal{F}(s)$ and $\mathcal{G}(s)$ dimensionless positive functions of $s = kh$ and $O(\lambda) =$ order of λ]

$$O(\partial_t^3 \xi_k) + \mathcal{F}(kh) \partial_t^2 \xi_k(t) + 2\nu k^2 \mathcal{G}(kh) \partial_t \xi_k(t) + \omega_k^2(t) \xi_k(t) = 0, \tag{10}$$

$$\mathcal{F}(s) = \tanh s [3 \cosh^2 s (\sinh 2s - 2s - 4s^3/3) + s^2(\sinh 2s - 2s)]/(\sinh 2s - 2s)^2,$$

$$\mathcal{G}(s) = \tanh s (\cosh 2s + 2s^2 + 1)/(\sinh 2s - 2s).$$

We can check that $O(\partial_t^3 \xi_k)/\mathcal{F}(kh)\partial_t^2 \xi_k \approx \Omega \ell^2/\nu = \Omega \tau_D$ and then (10) is an expansion in powers of $\Omega \tau_D$, and for $\Omega \tau_D \ll 1$ we conclude that the local second order equation (10) [without $O(\partial_t^3 \xi_k)$] is sufficient to describe the behavior of the surface. This condition can be written as $(\nu/\Omega)^{1/2} \gg \ell$. If we call δ the length of the boundary layer it is well known that in the case of weak viscosity $\delta = (\nu/\Omega)^{1/2}$, but, when viscosity increases, δ can saturate all the region in which the fluid is in motion, i.e., one has $\delta \approx \ell$, and the condition $(\nu/\Omega)^{1/2} \gg \ell$ means just that we are in this situation. This is what characterizes the lubrication regime and what we understand by high viscosity.

Since one always has $\ell < h$ we see that $\Omega \ll \nu/h^2$ is a sufficient condition to be in the lubrication regime. We now apply Eq. (10) to Faraday's instability, and in order to compare with other works we choose $g_e(t) = g(1 + \Gamma \cos \Omega t)$. The regime of weak viscosity for this forcing has been exhaustively studied by Kumar [2] who concluded that the instability is subharmonic. This means that the system oscillates at half the frequency of the driving force, and the mechanism of selection of the wavelength is the parametric resonance condition $\omega_k \sim \Omega/2$ (from now on, ω_k is the frequency of surface waves for constant g). Other frequencies are also resonant if $\omega_k \sim n\Omega/2$ with $n = 2, 3, 4 \ldots$ and to each new resonance we can associate a tongue in the space $\Gamma - k$, but these tongues have a higher threshold Γ since they are more strongly dissipated. In his work Kumar made only a brief incursion in the lubrication regime, reporting the appearance of a series of bicritical points in which a subharmonic region has the same threshold as harmonic region. We reproduce exactly these observations (see Fig. 1) with Eq. (10), and we can interpret them as the appearance of a new selection mechanism which is different from the usual one. Putting $\Psi = \xi_k e^{\gamma t}$ and $x = \Omega t$ in (10),

$$\Psi''(x) + \frac{1}{\Omega^2}[E - V(x)]\Psi(x) = 0\,, \qquad (11)$$

where $E \equiv \Omega_0^2 - \gamma^2$, $V(x) \equiv -V_0 \cos x$, $\Omega_0^2 \equiv \omega_k^2/\mathcal{F}(kh)$, $\gamma \equiv \nu k^2 \mathcal{G}(kh)/\mathcal{F}(kh)$, and $V_0 \equiv \Gamma\Omega_0^2/(1 + \tau k^2/\rho g)$. Notice that if (10) is the equation of a pendulum, Ω_0 is its frequency and 2γ its dissipation. Equation (11) is a Schrödinger-type equation in a periodic potential and we can use standard methods. The first thing we do is to follow the minimal threshold of each tongue when Ω is varying [see Fig. 2(a)]: A series of bicritical points appear, and this process saturates when $\Omega \to 0$ in a value Γ_*. If we now draw the most unstable mode corresponding to each Ω we obtain Fig. 2(b). The points of discontinuity of the curve arise each time one arrives at a bicritical point in Fig. 2(a), and we now observe a saturation of the most unstable mode at a value k_* when $\Omega \to 0$.

The case Ω small gives qualitative information of what is happening in the lubrication regime, and we can study this limiting case using Wentzel-Kramers-Brillouin (WKB) techniques in (11). We easily show that for $|E| > V_0$ the system is always stable. For $|E| < V_0$ we can calculate the amplification factor of the Floquet solutions which satisfy the relation $\xi_k(t + 2\pi/\Omega) = e^{\mu(2\pi/\Omega)}\xi_k(t)$, where μ is the Floquet exponent. The WKB calculation gives for the exponent

$$\mathrm{Re}(\mu) = \frac{V_0^{1/2}}{\pi}\int_{E/V_0}^{1}\left(\frac{x - E/V_0}{1 - x^2}\right)^{1/2} dx - \gamma\,. \qquad (12)$$

Since under the condition $|E| < V_0$ the integral is of order $O(1)$, we can estimate $\mathrm{Re}(\mu) \approx V_0^{1/2} - \gamma$, and the instability is then a competition the amplifying effect represented by $V_0^{1/2}$ and the dissipative effect represented by γ. We consider the dependence in k of the Floquet exponent, since $V_0^{1/2} \sim \Omega_0$ (except for a slowly varying function of k), an appropriate function to compare amplification, and dissipation is the ratio γ/Ω_0 (see Fig. 3): For big k the dissipative effect is dominant and those modes are very stable, while for small k the amplification is less effective in front of a dissipation which saturates $[\gamma(kh \ll 1) \approx 5\nu/4h^2]$. This provides an explanation for the appearance of an intermediate mode k_* which makes the amplification more effective and at the same time diminishes the dissipation. This mode k_* allows one to understand why the system goes through a series of bicritical points: Since the unstable modes are observed by the resonance condition $\omega_k \sim n\Omega/2$ (we are obviously abusing this condition which is correct for weakly viscous fluids), when Ω becomes smaller more and more tongues will have their characteristic mode k smaller than k_* and will not be favored while the tongues with $k \approx k_*$ will be privileged. From (12) the marginal curve $\mathrm{Re}(\mu) = 0$ is $\Gamma = \Delta(\gamma/\Omega_0)(1 + \tau k^2/\rho g)$, where $\Delta(s)$ is a dimensionless function defined for $s \geq 0$ with the properties $\Delta(0) = 1$, $\Delta'(0) = 2\sqrt{2}$, $\Delta' > 0$, $\Delta'' > 0$, and $\Delta(s > 1) \approx 4.33s^2 + 3.21$. The marginal curve can be written as $\Gamma = \tilde{\Gamma}(kh, \alpha, \beta)$ in terms of the adimensional variables kh, $\alpha \equiv \nu^2/h^3 g$ and $\beta \equiv \tau/\rho h^2 g$, and one can show that $\tilde{\Gamma}$ as a function of k has only one minimum $\Gamma_* = \tilde{\Gamma}(k_* h, \alpha, \beta)$ which corresponds (Fig. 3) to the saturation point (Γ_*, k_*). One necessarily has $\Gamma_* = \phi(\alpha, \beta)$, $k_* = \theta(\alpha, \beta)/h$, where ϕ and θ are dimensionless functions with the properties $\phi \geq 1$, $\phi(\alpha, 0) = \Delta(3.48\sqrt{\alpha})$ and $\theta \leq 1$, $\theta(\alpha, 0) = 0.86$.

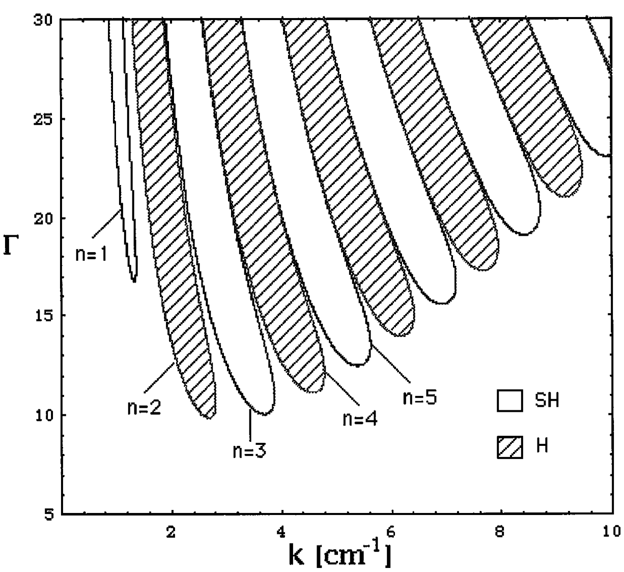

FIG. 1. Simulation of (11) for the same values of Kumar in Ref. [2]; $h = 0.2$ cm, $\nu = 1.02$ cm^2/s, $\rho = 1.2$ g/cm^3, $\tau = 67.6$ g/s^2, $\Omega/2\pi = 6$ Hz. The tongues are classified by the resonance condition, $n = 1$ is the first subharmonic tongue (SH), $n = 2$ is the following harmonic tongue (H), etc.

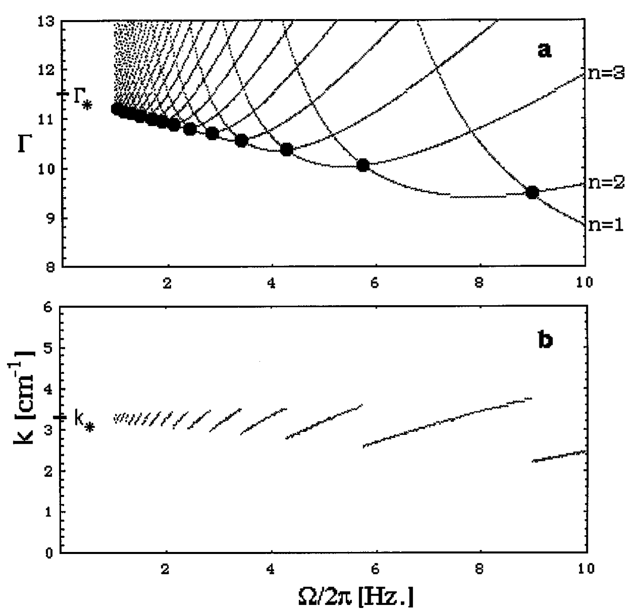

FIG. 2. (a) The minimum threshold of each tongue when the frequency of the driving force is changing. The black circles are the bicritical points. (b) The most unstable mode vs the value of the frequency of the driving force. The simulation is made for the same values of Fig. 1.

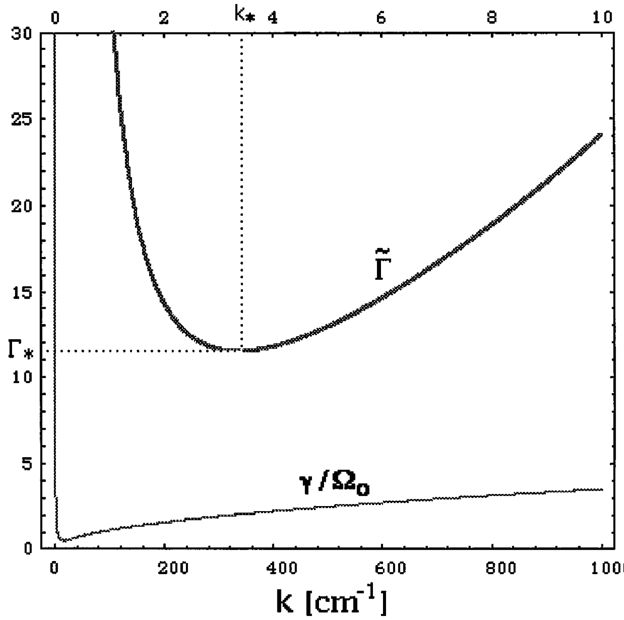

FIG. 3. The curve above is the marginal stability curve; its horizontal scale is in the top of the drawing. The curve beneath represents γ/Ω_0; its horizontal scale is in the bottom of the drawing.

It can be shown that $\theta(\alpha, \beta)$ is a slowly varying function of β and $\theta(\alpha > \beta, \beta) \approx 0.86$; consequently, in a wide range of parameters, the structure's wavelength is

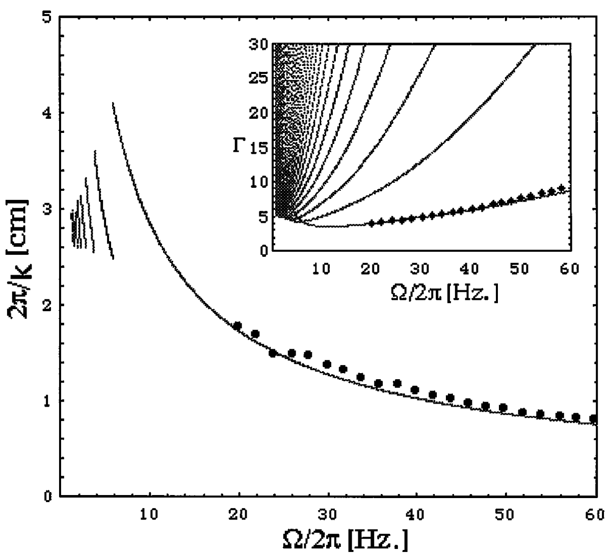

FIG. 4. The black points are experimental data for the wavelength and the threshold extracted from Ref. [2], and the curves are obtained by solving numerically the second order equation (11). The parameters are $h = 0.29$ cm, and (ν, ρ, τ) are the same as in Fig. 1.

determined by h. We can estimate in a very simple way the wavelength $2\pi/k_* \approx 2\pi h/0.86$ and the forcing $\Gamma_* \approx \Delta(3.48\sqrt{\alpha}) \approx 52.4\alpha + 3.21$, and the result agrees very well with the experimental data known to us as can be seen in Fig. 4 where our low frequency estimations give $2\pi/k_* \approx 2.1$ cm and $\Gamma_* \approx 4$. Studying the WKB solutions, we can verify that the amplification of the deformation of the surface occurs in the regions where $E - V(x) < 0$ (in the case $|E| < V_0$), where an effective negative acceleration generally dominates. The reason for the amplification is simple: The system is exposed to a negative acceleration and is unstable as the Rayleigh-Taylor instability [10], where a heavy fluid is over a light one. When $E - V(x) > 0$, the solutions change their phase and are damped. During a period both effects [amplification when $E - V(x) < 0$ and dissipation when $E - V(x) > 0$] compete, and relation (12) is the representation of this balance. This is qualitatively different from the usual parametric resonance mechanism for weakly viscous fluids since the origin of the amplification is different. As a final remark, we have explored here the low frequency region where the lubrication approximation is valid, but this does not exclude the validity of the approximation in other regions of the space of parameters which are being investigated, and this fact can explain the surprising agreement obtained with experimental results at much higher frequencies (Fig. 4).

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