Very Simple Proof of the Causal Propagation of Gravity in Vacuum

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In this Letter we present a new property of the Bel-Robinson tensor which allows us to give a very simple proof of the causal propagation of gravity in vacuum and that, moreover, provides an invariant characterization for Petrov type N space-times. [S0031-9007(96)02250-8]

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Causal propagation of gravity in vacuum follows from the Cauchy problem for the Einstein field equations in empty space-time (see [1], and references therein). Therefore, this result can be viewed as a consequence of the hyperbolic form of vacuum Einstein's equations. Here we present a new and very simple proof which does not make explicit use of this fact. It is just a consequence of the geometric properties of vacuum space-times. Particularly, it follows from what we have called the "dominant superenergy property" of the Bel-Robinson tensor. This property gives also a new characterization for Petrov type N space-times which makes a clear difference with the other two usually considered radiative types (II and III) (see, for instance, [2,3]).

To begin with, let us remember some properties of the Bel-Robinson tensor for any space-time [4] (see also [5]), whose definition is

$$\mathcal{T}^{\alpha\beta\lambda\mu} \equiv C^{\alpha\rho\lambda\sigma}C^{\beta\ \ \mu}_{\ \rho\ \ \sigma} + C^{*\alpha\rho\lambda\sigma}C^{*\beta\ \ \mu}_{\ \ \rho\ \ \sigma},$$

where $C_{\alpha\beta\lambda\mu}$ is the Weyl tensor and "*" is the usual dual operation. The Bel-Robinson tensor is completely symmetric and traceless:

$$\mathcal{T}^{\alpha\beta\lambda\mu} = \mathcal{T}^{(\alpha\beta\lambda\mu)}, \qquad \mathcal{T}^{\alpha\lambda\mu}_{\alpha} = 0,$$

and it is covariantly conserved in empty space-time (with or without a cosmological constant Λ), that is to say,

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \Longrightarrow \nabla_\alpha \mathcal{T}^{\alpha\beta\lambda\mu} = 0, \tag{1}$$

where $R_{\alpha\beta}$ is the Ricci tensor. The "superenergy" density relative to any observer described by the timelike unit vector field $\vec{u}$, $u_\mu u^\mu = -1$, is defined by

$$W(\vec{u}) \equiv \mathcal{T}^{\alpha\beta\lambda\mu}u_\alpha u_\beta u_\lambda u_\mu \geq 0, \tag{2}$$

so that it is non-negative and satisfies the following fundamental property:

$$\exists\ \vec{u}, \qquad W(\vec{u}) = 0 \Longleftrightarrow C_{\alpha\beta\lambda\mu} = 0 \Longleftrightarrow \mathcal{T}_{\alpha\beta\lambda\mu} = 0.$$

Let us now define, for any timelike unit vector field $\hat{u}$, the following vector:

$$\mathcal{T}^{\alpha}(\hat{u}) \equiv \mathcal{T}^{\alpha\beta\lambda\mu}u_\beta u_\lambda u_\mu,$$

which is analogous to the typical local energy-flow vector $T_{\alpha\beta}u^\beta$ constructed with the usual energy-momentum tensor $T_{\alpha\beta}$. By introducing the well-known electric and magnetic parts of the Weyl tensor [6,7]

$$E_{\alpha\lambda} \equiv C_{\alpha\beta\lambda\mu}u^\beta u^\mu, \qquad H_{\alpha\lambda} \equiv -C^*_{\alpha\beta\lambda\mu}u^\beta u^\mu,$$

which are symmetric, traceless, and spatial (that is to say, orthogonal to $\vec{u}$), $\mathcal{T}^\alpha(\vec{u})$ and $W(\vec{u})$ can be expressed as

$$\begin{aligned} \mathcal{T}^\alpha(\vec{u}) &= -W(\vec{u})u^\alpha + 2\eta^{\alpha\beta\lambda\mu}E_{\beta\sigma}H^\sigma_\lambda u_\mu, \\ W(\vec{u}) &= E_{\rho\sigma}E^{\rho\sigma} + H_{\rho\sigma}H^{\rho\sigma}. \end{aligned} \tag{3}$$

Then we have the following result.

*Proposition 1.—$\mathcal{T}^\alpha(\vec{u})$ is nonspacelike (*i.e., $\mathcal{T}^\alpha\mathcal{T}_\alpha \leq 0$*), for all $\vec{u}$, $u_\mu u^\mu = -1$.*

The proof is as follows. Given any two spatial tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$, we will denote by AB their matrix product; that is to say, AB is the spatial tensor with components $(AB)_{\alpha\beta} \equiv A_{\alpha\sigma}B^\sigma_\beta$. We also define the following *positive-definite* inner product $A \cdot B \equiv A_{\alpha\beta}B^{\alpha\beta}$. A direct calculation gives

$$\begin{aligned} \mathcal{T}^\alpha\mathcal{T}_\alpha &= -W^2 + 4(EH)\cdot(EH - HE) \\ &= -W^2 - 2(EH + HE)\cdot(EH + HE) \\ &\quad + 8(EE)\cdot(HH). \end{aligned}$$

Then, by using the Cauchy-Schwarz inequality $(A \cdot B)^2 \leq (A \cdot A)(B \cdot B)$ applied to $A = EE$ and $B = HH$ we get

$$\begin{aligned} \mathcal{T}^\alpha\mathcal{T}_\alpha &\leq -W^2 - 2(EH + HE)\cdot(EH + HE) + 8[(EE)\cdot(EE)]^{1/2}[(HH)\cdot(HH)]^{1/2} \\ &= -(E\cdot E + H\cdot H)^2 - 2(EH + HE)\cdot(EH + HE) + 8[(EE)\cdot(EE)]^{1/2}[(HH)\cdot(HH)]^{1/2}, \end{aligned}$$

where we have used (3) to obtain the second expression. Now, for any spatial traceless symmetric tensor $A_{\alpha\beta}$, it can be shown that

$$(A\cdot A)^2 = 2[(AA)\cdot(AA)], \tag{4}$$

and, therefore, we finally obtain

$$\begin{aligned} \mathcal{T}^\alpha\mathcal{T}_\alpha &\leq -(E\cdot E - H\cdot H)^2 \\ &\quad - 2(EH + HE)\cdot(EH + HE) \leq 0, \end{aligned}$$

which proves the proposition.

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This property together with (2) constitute what we call the dominant superenergy property for the Bel-Robinson tensor, in analogy with the dominant energy condition for the energy-momentum tensor (see [8]). An interesting corollary is the following.

Corollary.—*$\exists\ \vec{u}$, $u_\mu u^\mu = -1$, such that $\mathcal{T}^\alpha \mathcal{T}_\alpha = 0$ with $\mathcal{T}^\alpha \neq 0$ if and only if the space-time is of Petrov type N. On the other hand, $\exists\ \vec{u}$, $u_\mu u^\mu = -1$, such that $\mathcal{T}^\alpha = 0$ if and only if the Petrov type is O.*

The Petrov type is O if and only if (iff) $\mathcal{T}^{\alpha\beta\lambda\mu} = 0$, so that this is equivalent to $\mathcal{T}^\alpha = 0$. If the Petrov type is N then, as is well known, the Bel-Robinson tensor takes the very simple form $\mathcal{T}_{\alpha\beta\lambda\mu} = l_\alpha l_\beta l_\lambda l_\mu$, where l_μ points along the unique multiple null direction of the Weyl tensor (see, for instance, [2,3,5,9]), from which it is obvious that $\mathcal{T}^\alpha(\vec{u})$ is always a nonzero null vector. Conversely, whenever $\mathcal{T}^\alpha \mathcal{T}_\alpha = 0$, and taking into account that the equality in the Cauchy-Schwarz inequality holds iff $A = \lambda B$, we can conclude that $EE - HH = 0$ and $EH + HE = 0$. But this is equivalent to having $(E + iH)(E + iH) = 0$, which is one of the characterizations for Petrov type N (or type O in the case $E = H = 0$). (For this and other simple matters regarding the Petrov classification, see, for instance, [2,6].)

The previous result is therefore an intrinsic characterization for Petrov type N space-times. Moreover, the analogy with the electromagnetic field [9] (where the local energy-flow vector $T_{\alpha\beta}u^\beta$ is a null vector iff pure electromagnetic radiation exists) provides a possible criterion for the definition of intrinsic states of gravitational radiation. This criterion is in accordance with that of Lichnerowicz (see [2,3,9]). There are, however, other similar but different criteria in the literature, see [2,3], and it is not clear to us as yet which is the most appropriate one. In any case, we believe that the above proposition may serve, at least, to refine any possible characterization of intrinsic gravitational radiation by distinguishing between the type N and the other two usually considered radiative Petrov types (II and III) [2,3].

Let us now proof another result which will be needed later. To that end, let us define, for any unit timelike vector $\vec{u}$, the spatial symmetric tensor

$$\mathcal{T}^{\alpha\beta}(\vec{u}) \equiv \mathcal{T}^{\sigma\rho\lambda\mu} P^\alpha_\sigma P^\beta_\rho u_\lambda u_\mu \,,$$

where $P^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta$ is the projector orthogonal to $\vec{u}$.

Proposition 2.—*For all unit $\vec{u}$ we have $\mathcal{T} \cdot \mathcal{T} \leq W^2$.*

In order to prove the proposition let us compute explicitly the tensor $\mathcal{T}^{\alpha\beta}(\vec{u})$

$$\mathcal{T}^{\alpha\beta}(\vec{u}) = W(\vec{u})P^{\alpha\beta} - 2(E^\alpha_\sigma E^{\beta\sigma} + H^\alpha_\sigma H^{\beta\sigma}) \,,$$

so that we have

$$\mathcal{T}^{\alpha\beta}\mathcal{T}_{\alpha\beta} = -W^2 + 4[(EE)\cdot(EE) + (HH)\cdot(HH) + 2(EE)\cdot(HH)] \,,$$

from where, using again (4) and the Cauchy-Schwarz inequality, we can easily get $\mathcal{T}^{\alpha\beta}\mathcal{T}_{\alpha\beta} \leq W^2$, as we wanted to show.

From propositions 1 and 2, we can deduce that in any orthonormal tetrad $\{\vec{u}, \vec{e}_i\}$ $(i = 1,2,3)$ (with $\vec{u}$ as the zeroth vector in the tetrad),

$$\mathcal{T}^{0000} \geq |\mathcal{T}^{\alpha\beta 00}| \,.$$

The above propositions and corollary hold obviously also in the case that $\vec{u}$ is a timelike but non-necessarily unit vector.

The above results allow us to prove in a very simple way the causal propagation of gravity in vacuum. The proof is analogous to that of the "conservation theorem" for the matter fields, which can be found in [8]. From now on we will consider only empty space-time, with or without a cosmological constant Λ, so that $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$. Let us take any compact region of space-time $\mathcal{K}$ with boundary $\partial\mathcal{K}$ (see Fig. 1) in which the stable causality condition (see [8]) holds. Then, $\mathcal{K}$ can be foliated by spacelike hypersurfaces $\Sigma_t \equiv \{t = \text{const}\}$, where t is a time function whose gradient $\mathbf{v} = \mathbf{dt}$ is timelike everywhere on $\mathcal{K}$. The boundary $\partial\mathcal{K}$ is divided into three parts: $(\partial\mathcal{K})_1$ and $(\partial\mathcal{K})_2$ are the past and future nontimelike boundaries, respectively, and $(\partial\mathcal{K})_3$ is the remaining part, which may be empty (see Fig. 1). Suppose now that $C_{\alpha\beta\lambda\mu} = 0$ on $(\partial\mathcal{K})_1$ and $(\partial\mathcal{K})_3$, and define the following "superenergy integral" (which is a non-negative function of t)

$$w(t) \equiv \int_{J^-(\Sigma_t)\cap\mathcal{K}} W(\vec{v})\boldsymbol{\eta} = \int^t \left(\int_{\Sigma_{t'}\cap\mathcal{K}} \mathcal{T}^\alpha(\vec{v})\, d\sigma_\alpha|_{\Sigma_{t'}} \right) dt' \geq 0 \,,$$

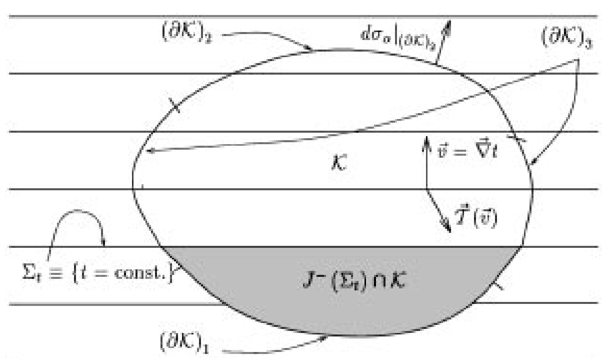

FIG. 1. Compact region $\mathcal{K}$ with boundary $\partial\mathcal{K}$. As usual, light cones are at $\pm 45°$ and the future direction points upwards. Then, the boundary $\partial\mathcal{K}$ has three different parts: the past and future nontimelike parts, which are marked as $(\partial\mathcal{K})_1$ and $(\partial\mathcal{K})_2$, respectively, and the remaining part denoted by $(\partial\mathcal{K})_3$. $\mathcal{K}$ is foliated by spacelike hypersurfaces $\Sigma_t \equiv \{t = \text{const}\}$, where t is a time function. The shadowed zone corresponds to the causal past of the hypersurface Σ_t in $\mathcal{K}$, that is to say, to $J^-(\Sigma_t) \cap \mathcal{K}$.

where as usual $J^{-}(\Sigma_t)$ denotes the causal past of Σ_t, $\boldsymbol{\eta}$ is the canonical volume 4-form and $d\sigma_\alpha|_{\Sigma_{t'}}$ is the hypersurface element of $\Sigma_{t'}$, which points along $\vec{v}$. Then, by using the Gauss theorem, we arrive at

$$\begin{aligned}\frac{dw}{dt} &= \int_{\Sigma_t \cap \mathcal{K}} \mathcal{T}^{\alpha}(\vec{v})\, d\sigma_\alpha|_{\Sigma_t} \\ &= \int_{J^{-}(\Sigma_t)\cap\mathcal{K}} \nabla_\alpha \mathcal{T}^{\alpha}(\vec{v})\boldsymbol{\eta} - \int_{J^{-}(\Sigma_t)\cap(\partial\mathcal{K})_2} \\ &\quad \times \mathcal{T}^{\alpha}(\vec{v})\, d\sigma_\alpha|_{(\partial\mathcal{K})_2}\,.\end{aligned}$$

The dominant superenergy property tells us that $\mathcal{T}^{\alpha}(\vec{v})\, d\sigma_\alpha|_{(\partial\mathcal{K})_2}$ is non-negative and, on the other hand, proposition 2 implies the existence of some constant $M > 0$ such that $\mathcal{T}^{\alpha\beta\lambda\mu}(\nabla_\alpha\nabla_\beta t)\nabla_\lambda t\nabla_\mu t \le (M/3)W(\vec{v})$ on the compact $\mathcal{K}$ (where the components of $\nabla_\alpha t$ and $\nabla_\alpha\nabla_\beta t$ are bounded). Thus, by taking into account (1), we have

$$\begin{aligned}0 \le \frac{dw}{dt} \le 3\int_{J^{-}(\Sigma_t)\cap\mathcal{K}} \mathcal{T}^{\alpha\beta\lambda\mu}(\nabla_\alpha\nabla_\beta t) \\ \times \nabla_\lambda t\nabla_\mu t\boldsymbol{\eta} \le Mw\,.\end{aligned}$$

From this, and given that $w(t)$ vanishes for early enough values of t, it follows that $w(t)$ will vanish for all t, which implies that $W(\vec{v}) = 0 \Longleftrightarrow \mathcal{T}_{\alpha\beta\lambda\mu} = 0$ (or, equivalently $C_{\alpha\beta\lambda\mu} = 0$) on $\mathcal{K}$. Thus, analogously to the case of matter fields, we have the following:

The conservation theorem.—In empty space-time (with a possible cosmological constant), if the Weyl tensor (or, equivalently, the Bel-Robinson tensor) is zero on $(\partial\mathcal{K})_3$ and on the initial hypersurface $(\partial\mathcal{K})_1$, then it is zero everywhere on $\mathcal{K}$.

From this theorem it can be deduced that, in vacuum, if the Weyl tensor is zero on a closed achronal set S then it is zero on its future Cauchy development $\mathcal{D}^{+}(S)$ [8] (see Fig. 2). This follows because int$[\mathcal{D}^{+}(S)]$ is globally hyperbolic (and thus causally stable) for achronal S, and then we can take its closure $\overline{\mathcal{D}^{+}(S)}$ as the compact $\mathcal{K}$ to apply the above theorem (see Fig. 2). It is obvious that this result holds equally for the past Cauchy development $\mathcal{D}^{-}(S)$ of S. This result is then interpreted as saying that gravity propagates causally in vacuum, in the sense that it cannot travel faster than light.

Some important remarks are in order. First, we wish to stress that in the case under consideration ($R_{\mu\nu} = \Lambda g_{\mu\nu}$), the full Riemann tensor can be written as

$$R_{\alpha\beta\lambda\mu} = C_{\alpha\beta\lambda\mu} + \frac{\Lambda}{3}(g_{\alpha\lambda}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\lambda})\,.$$

Thus, the cosmological constant gives only a kind of "background" constant curvature which does not propagate at all. Therefore, we have shown the causal propagation of the part of the curvature tensor that *can* propagate in vacuum.

Second, it is interesting to notice that, contrary to what happened in the case of matter fields where the

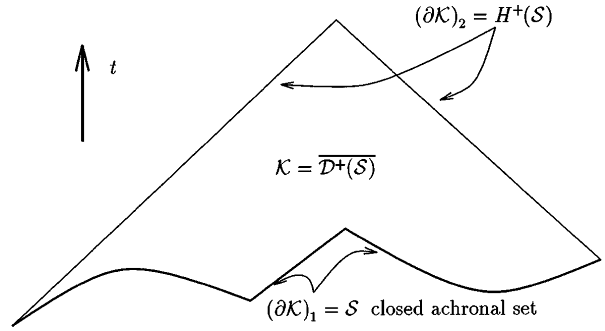

FIG. 2. The particular case in which the compact region $\mathcal{K}$ of Fig. 1 is the closure of the future Cauchy development $\mathcal{D}^{+}(S)$ of any closed achronal set S. As in Fig. 1, light cones are at $\pm 45°$. As we see, now $(\partial\mathcal{K})_1$ is the closed achronal set S itself, $(\partial\mathcal{K})_2$ is the future Cauchy horizon $H^{+}(S)$ of S (which is null), and $(\partial\mathcal{K})_3$ is empty.

dominant energy condition *must be assumed* in order to assure the causal propagation of matter [8], in our case the dominant superenergy property does not have to be imposed: It is just a property which the gravitational field has. Of course, this might depend on the particular theory describing the gravitational field (we have used Einstein's general relativity), but in any of these possible theories the result would hold if there is a superenergy tensor with the properties of the Bel-Robinson tensor. This will be true for most theories describing gravity in a geometrical way and with appropriate vacuum field equations.

Finally, it is curious that the proof we have presented for the causal propagation of gravity in vacuum is purely geometric and does not seem to be a consequence of the hyperbolic character of the vacuum Einstein field equations (even though we have made use of them). Furthermore, our result reinforces the fact that the Bel-Robinson tensor is a very useful tool in proving some global properties of space-times, as has been already manifested in the proof of the uniform asymptotic behavior of solutions to linear field equations in Minkowski space-time [10].

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