Helical Plasma Confinement Devices with Good Confinement Properties

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The criterion of approximate omnigeneity (i.e., having bounce-averaged drift lying within the magnetic surfaces) is much easier to satisfy than quasihelicity, the condition that B, the magnitude of the magnetic field, is a function of only a single linear combination of the toroidal angles. Simple criteria for omnigeneity are presented and used to construct exactly omnigenous forms for B that are far from quasihelical. Though this construction gives a nonanalytic function B, close to the constructed systems there exist other systems with analytic B. These results indicate that finding helical plasma confinement systems with minimal neoclassical transport is much easier than previously believed. [S0031-9007(96)02275-2]

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Stellarators having good transport properties would be ideal plasma confinement devices, as with externally generated rotational transform, they could be run in steady state without complicated current drive schemes, and stellarators are not susceptible to disruptions. Unfortunately, stellarators as originally proposed had large particle loss rates due to the fact that they contained trajectories (those locally trapped in a helical magnetic well) that directly drift out of the machine. This has led to research (e.g., Ref. [1]) geared towards finding configurations lacking such particles. Nearly a decade ago, Nuhrenberg and Zille [2] proposed that stellarators be quasihelical (termed quasisymmetric in recent work [3]), which means that the magnetic strength B is a function of only a single linear combination of the toroidal angles in Boozer [4] coordinates. They were able to find large-aspect-ratio quasihelical systems. Subsequently many researchers began asking whether smaller aspect ratio systems, which would be more compact, exist. This was answered by the work of Garren and Boozer [5], who found that the condition of quasihelicity cannot be satisfied beyond a certain order in an expansion in the distance from the magnetic axis. Thus, at this stage it would seem that small-aspect-ratio stellarators having good transport properties would not exist.

In this Letter we show that there remains hope for finding stellarator confinement systems with good transport at low aspect ratio. (Many of the details of our discussion have been relegated to a longer article [6].) We prove that omnigenous systems [7], those for which the bounce averaged drift remains within a flux surface, form a larger class than quasihelical systems. In this proof we find a precise condition for omnigeneity—that the contours of magnetic strength $|B|$ on a magnetic surface have constant angular separation in Boozer coordinates. However, the situation is more complicated upon closer examination, as we show (1) that omnigenous systems for which the magnetic strength is an analytic function must be quasihelical, yet (2) that one can have systems with analytic magnetic strength functions that are far from quasihelical while very nearly omnigenous. This last result indicates that in a practical sense even analytic omnigenous systems form a larger class than quasihelical systems. Finally, we propose simple design criteria for systems with good confinement properties towards the end of this Letter.

Our results are related to those of Ref. [3]. In Ref. [3] it was noted that in isometric systems, those for which the magnetic contours within a surface are separated by constant distance along a magnetic field line, the trajectories are omnigenous. Our condition of constant angular separation turns out to be the same. Thus, of our above results, that most closely related to Ref. [3] is to show that isometry is not only sufficient, but also necessary. This means that there is now a precise condition for omnigenous systems.

Our results are most easily arrived at in Boozer coordinates, which (ψ, θ, φ) are a special form of flux variables (in which magnetic field lines are linear in the angles) in that the covariant angular components of the magnetic field are constant. Because they are flux coordinates, the magnetic field has the Clebsch representation, $\mathbf{B} = \nabla\psi \times \nabla\theta + \iota(\psi)\nabla\varphi \times \nabla\psi$. This implies that the vector potential has the form, $\mathbf{A} = \psi\nabla\theta + A_\varphi(\psi)\nabla\varphi$, where $\iota = -dA_\varphi/d\psi$. As noted by Boozer, these angles can be further specified by requiring that the angular covariant components of the magnetic field be flux functions, $\mathbf{B} = B_\psi(\psi,\theta,\varphi)\nabla\psi + B_\theta(\psi)\nabla\theta + B_\varphi(\psi)\nabla\varphi$.

In these coordinates, Littlejohn's guiding-center Lagrangian [8] has the form

$$L_{\mathrm{gc}} = (muB_\psi/B)\dot{\psi} + (e\psi/c + muB_\theta/B)\dot{\theta} + (eA_\varphi/c + muB_\varphi/B)\dot{\varphi} - h\,, \tag{1}$$

where

$$h = \tfrac{1}{2}mu^2 + \mu B + e\Phi \tag{2}$$

is the Hamiltonian, u is the parallel velocity, and $\mu = \frac{1}{2}mv_\perp^2/B$ is the magnetic moment. The Euler-Lagrange equations for this Lagrangian give the guiding-center equations of motion. This is a phase-space Lagrangian, which is to say that the resulting equations of motion are

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first-order differential equations, and there is no further transformation to a Hamiltonian. Indeed, Darboux's theorem guarantees the existence of a transformation from the physical guiding-center variables $(\mathbf{X}, u)$ to local canonical variables. (Canonical formulations of guiding-center dynamics were developed [9] independently of the guiding-center Lagrangian.)

As already noted, quasihelical systems are those for which $|B|$ is a function of only a single linear combination of the angles, say $\varsigma \equiv N\varphi - \ell\theta$, where N is the toroidal mode number and ℓ is the poloidal mode number of the dominant Fourier component of the magnetic field strength. One can think of this helical angle ζ as roughly constant if one were to remain under a given coil in a helical coil system. [From near-axis analysis it follows that $\ell = 1$: Only $\ell = 1$ and $\ell = 2$ helical ($N \neq 0$) fields produce nonzero rotational transform at the magnetic axis. On axis the pressure gradient vanishes, so B must have nonvanishing gradient in order to balance magnetic curvature in the equation, $\mathbf{j} \times \mathbf{B} = \kappa B^2 - \frac{1}{2}\nabla_\perp B^2 = 0$. Hence, $\ell = 1$ fields must be present. Thus, if only a single helicity is present, it must be $\ell = 1$ and $N \neq 0$.] For such systems, it is useful to transform the Lagrangian (1) to the angles (θ, ζ). This gives

$$L_{\text{gc}} = \left(\frac{muB_\psi}{B}\right)\dot{\psi} + \left[\frac{e}{c}\left(\psi + \frac{A_\varphi}{N}\right) + \frac{mu}{B}\left(B_\theta + \frac{B_\varphi}{N}\right)\right]\dot{\theta} + \left(\frac{e}{c}\frac{A_\varphi}{N} + \frac{mu}{B}\frac{B_\varphi}{N}\right)\dot{\zeta} - h\,. \tag{3}$$

The last important fact is that the covariant component B_ψ contains the same helicities as B in MHD equilibrium [4]. In the present case this implies that the Lagrangian (3) is independent of θ. Hence, from the Euler-Lagrange equations it follows that

$$P_h \equiv \frac{\partial L_{\text{gc}}}{\partial\dot{\theta}} = \frac{e}{c}\left(\psi + \frac{A_\varphi}{N}\right) + \frac{mu}{B}\left(B_\theta + \frac{B_\varphi}{N}\right) \tag{4}$$

is an invariant of the motion. Analogous to the axisymmetric case, the existence of the invariant (4) guarantees orbits with good properties. Energy conservation guarantees that the variation of the parallel velocity and, hence, the final term in Eq. (4) is bounded. Thus, the variation of $\psi + A_\varphi/N$ is small, and so orbits remain confined to the vicinity of a flux surface.

A less restrictive way to achieve systems with good trajectories is to require omnigeneity [7], the property whereby the bounce-averaged cross-flux-surface drift vanishes. As the bounce-averaged drift conserves the bounce (or longitudinal) action through lowest order in the expansion in the drift frequency relative to the bounce frequency, this implies that the bounce action is constant on a surface. Thus, we need to find systems where the bounce action, or zeroth-order bounce adiabatic invariant,

$$J_0 = \oint mu\hat{b}\cdot d\mathbf{r} = m(\iota B_\theta + B_\varphi)\oint (u\,d\varphi/B)\,, \tag{5}$$

where the velocity $u = \sqrt{2(E - \mu B - e\Phi)/m}$ is determined by energy conservation, and the loop integral is along a field line between reflection points, $E = \mu B$, is constant on a magnetic surface. An immediate consequence of the condition that J is constant on a magnetic surface is the fact that the local minima of the magnetic field along field lines in a given surface have the same value of B. This is the principle behind the improved confinement for the systems in Ref. [1]. One can also show that the magnetic maxima and the action of particles at the trapped-passing boundary have the same value on a surface, and, hence, that transition orbits [10], which are chaotic due to separatrix crossing [11], are absent in omnigenous systems.

We can write the bounce action in the form

$$J_0 = 2\sqrt{2m}\,(\iota B_\theta + B_\varphi)\int_{B_{\min}}^{(E-e\Phi)/\mu}\frac{dB}{B} \times\sqrt{(E - \mu B - e\Phi)}\sum_{\pm}\left|\frac{d\varphi}{dB}\right|, \tag{6}$$

where

$$F(\psi, B, \varphi_0) \equiv \sum_{\pm}\left|\frac{d\varphi}{dB}\right| \tag{7}$$

is the sum of the change in toroidal angle with respect to magnetic strength at the two points having the same value of B on a given magnetic surface and on a given field line, labeled by $\varphi_0 \equiv \theta - \iota\varphi$. We have shown [6] that the integral transform in Eq. (6) is invertible. Thus, if J is independent of the field line (φ_0), then so is F. From this it follows that the angular separation,

$$\Delta\varphi = 2\int_{B_{\min}}^{B} dB'\,F(B')\,, \tag{8}$$

of any two contours of the same value of B is constant on a magnetic surface.

Equation (8) is a central result from which many consequences follow. First, it can be shown that the contour of the maxima is straight in Boozer coordinates provided the rotational transform is irrational. Second, one can construct omnigenous functions B that depend nontrivially on θ in addition to ζ by the following procedure. We introduce the new coordinate η such that the magnetic field has the form

$$B/B_0 = 1 + \varepsilon_r\cos(\eta)\,. \tag{9}$$

This is generally possible since for every value of θ, B varies from the same maximum value to the same minimum value as one moves in ζ. Next, we assume that we know the transformation to the left of the minimum for all the field lines and along one field line through the origin ($\theta = 0, \zeta = 0$) for one full period. That is

$$\begin{aligned}\zeta &= \eta + g(\theta, \eta) \quad \text{for } 0 \le \eta \le \pi\,,\\ \zeta &= q_h\theta \qquad \text{for } 0 \le \zeta \le 2\pi\,.\end{aligned} \tag{10}$$

Here, $q_h \equiv d\varsigma/d\theta = N/\iota - 1$ gives the rate of change of the helical angle with respect to the poloidal angle as one moves along a field line. The function $g(\theta, \eta)$ is chosen to vanish at $\eta = 0$ so that the curve of maximum is at $\zeta = 0$ and $\zeta = 2\pi$. From the knowledge of the transformation along the one field line segment, we can determine the angular difference $\Delta\zeta(B)$. One simply finds the two points having the given value of magnetic strength along the one segment and takes the difference to calculate $\Delta\zeta(B) = \Delta\zeta(\eta)$. Finally, we determine the transformation for $\eta > \pi$ by requiring the condition for omnigeneity,

$$\zeta = 2\pi - \eta + g(\theta - \Delta\zeta(\eta)/q_h, 2\pi - \eta) + \Delta\zeta(\eta) \quad \text{for } \eta > \pi . \tag{11}$$

This equation is interpreted as follows. To find ζ to the right of the minimum, one applies the transformation equation to the corresponding point $[\theta - \Delta\zeta(\eta)/q_h, \zeta - \Delta\zeta(\eta)]$ and the corresponding value $(2\pi - \eta)$ of the new angle. Equation (11) gives the transformation explicitly everywhere on the torus. It is straightforward to verify that this transformation has $\zeta = 2\pi$ at $\eta = 2\pi$. Thus, the solution is periodic. Finally, the transformation is one to one for sufficiently small g, as are all near-identity transformations.

Unfortunately, the above construction leads to nonanalytic transformations if g has nontrivial dependence on θ. We define the function $G(\theta, \eta)$ to give the transformation for all values of η by the formula

$$\zeta = \eta + G(\theta, \eta) . \tag{12}$$

Thus, $G = g$ for $0 < \eta < \pi$, and G follows from Eq. (11) for $\pi < \eta < 2\pi$. One can show from differentiation of Eq. (12) that G and all of its derivatives are constant on the contour of maxima. From this and analyticity it follows that G is independent of θ. Thus, analytic, omnigenous magnetic fields are quasihelical.

These results appear pessimistic. We expect the magnetic field strength to be an analytic function. Yet, if it is, the magnetic field must be quasihelical. This would seem to eliminate the possibility of having systems with good transport that are far from quasihelical. However, analyticity is a fragile concept. Two functions, one analytic and one not analytic, can be arbitrarily close in value everywhere. Hence, one must ask the different question: Can we find functions that are analytic, yet close everywhere to a nonanalytic function that is far from quasihelical? If so, then it is possible to have magnetic fields that are very far from quasihelical, yet very nearly omnigenous.

To test this idea, we used the above construction to obtain an omnigenous B. We chose $\varepsilon_r = 0.25$, $q_h = 8.809$, and the transformation

$$g(\theta, \eta) = [a_0 + a_s \sin(\theta) + a_c \cos(\theta)] \sin(\eta/2) , \tag{13}$$

with parameters $a_0 = 0.1$, $a_s = 0.8$, and $a_c = 0.05$. This transformation is not periodic. We chose this form in order to have $g(\theta, \pi)$ nonzero while having vanishing $g(\theta, 0)$. The fact that g is not periodic does not matter, as the magnetic field does not depend on g for $\eta > \pi$.

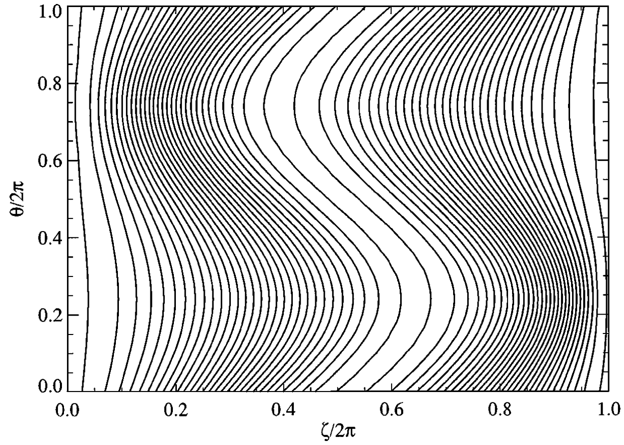

FIG. 1. Contours of the omnigenous magnetic field strength.

The contours of the magnetic field strength are shown in Fig. 1. That this field is not quasihelical is evidenced by the fact that the contours are not lines of constant ζ. As noted, this magnetic field strength is not analytic. Indeed, its second derivative is not continuous. This can be seen in the fact that the contours near $\zeta = 0$ and $\zeta = 2\pi$ are not symmetric about that line. Examination of this figure with a ruler shows that the contours have constant separation, as per Eq. (8), which implies that $\Delta\zeta$ is also constant.

To obtain an analytic magnetic field strength that is close to that shown in Fig. 2 we carried out the following operations. We Fourier analyzed the magnetic field, kept only up to the first two harmonics [up to $\cos(\zeta \pm 2\theta \pm 2\zeta)$], and transformed back. As the Fourier series is finite, this magnetic strength is analytic. In Fig. 2 we show the contours for this magnetic field. Some differences

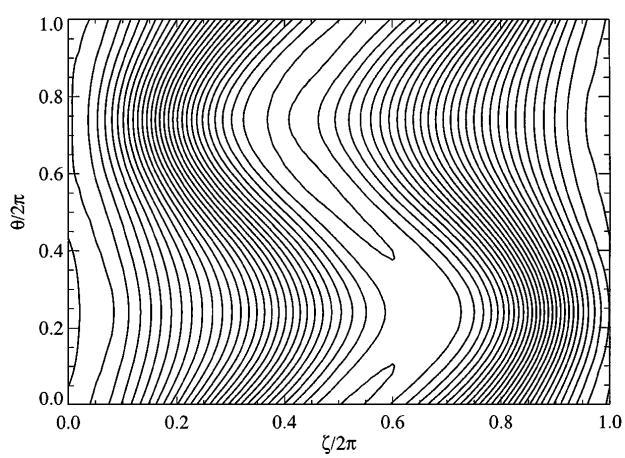

FIG. 2. Contours of the nearly omnigenous, analytic magnetic field, the magnetic field obtained by truncating the Fourier series of the omnigenous field through the first two harmonics in each angle.

are notable. For example, the curve of maxima is no longer straight, and the fact that there is a contour with the topology of a circle indicates that the minima no longer all have the same value, although they do not vary by much (about 2%).

To test the omnigeneity of these configurations, we integrated several trajectories in the analytic, nearly omnigenous magnetic field found above. The poloidal cross section of the trajectory of a typical locally trapped particle is shown in Fig. 3. [The coordinates are $x = r\cos(\theta)$ and $y = r\sin(\theta)$, where $r = \sqrt{\psi/\psi_{\rm edge}}$, and B_0 is the value of the magnetic field on axis. We have taken $\varepsilon_r = \delta_r r$ with $\delta_r = 0.25$.] We have set the dimensionless parameters to be $E/m\Omega_0^2 a^2 = 2 \times 10^{-3}$ and $\mu B_0/m\Omega_0^2 a^2 = 1.81 \times 10^{-3}$, where E is the energy, Ω_0 is the gyrofrequency on axis, and a is the minor radius. (These are the numbers for a 17 keV proton in a machine with 1 T magnetic field and minor radius of 30 cm.) This figure shows that the trajectory has remained, on average, close to its initial flux surface. (In the typical stellarator fields such a trajectory would intersect the wall.) One can see that this system is not quasihelical by the fact that the oscillation width varies with poloidal angle.

These results show that it is possible to have a magnetic field strength that is analytic, nearly omnigenous, but far from quasihelical. For such fields the wells are distorted, but the particles stay near a particular flux surface. Thus, the large neoclassical losses associated with trajectories that directly drift out of the plasma will not be present. Because our requirements for the magnetic strength are less restrictive than those for quasihelicity, there is hope of finding systems with good confinement properties at smaller aspect ratios.

Ultimately one would like to set certain limited criteria for the design of a stellarator, as one does not expect to obtain perfectly omnigenous systems. The condition proposed in Ref. [1] is that of constant magnetic minima, so that the deeply trapped particles are omnigenous. As noted above, the conditions of constant magnetic maxima and constant separatrix action of a surface ensure both omnigeneity of the marginally trapped particles and elimination of the chaotic transition particles. With all three conditions, the two extremes of locally trapped particles are omnigenous, and there are no chaotic trajectories. (As shown in Ref. [1], having only constant magnetic maxima may do little to reduce transport.) We suggest these as starting design criteria, though naturally one will have to check such systems to ensure that the trajectories of the intermediately trapped particles do not make excessive drift excursions.

We have noted that such systems should have much better neoclassical confinement properties than the usual helical confinement systems. Moreover, these systems should have better confinement than toroidal systems with equal magnetic field variation; the variation of the flux surface label is smaller than that in a tokamak by the ratio, $N/\iota - 1$, of connection lengths. Indeed, this leads to the interesting speculation that transport due to ballooning mode turbulence, which increases with connection length, will also be small in omnigenous helical systems.

It remains a significant area of research to find systems satisfying the criteria we have laid out. Such an undertaking will require the use of the large vacuum and equilibrium computer applications in use at major labs around the world. We suggest that if such studies are successful, one may end up with very good toroidal plasma confinement systems.

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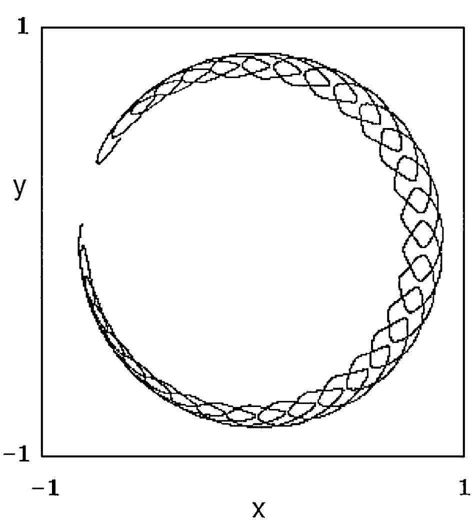

FIG. 3. Poloidal cross section of the trajectory of a typical locally trapped particle in the nearly omnigenous magnetic field.

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