Failure of the Quasimonochromatic Approximation for Ultrashort Pulse Propagation in a Dispersive, Attenuative Medium

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The dynamical evolution of an ultrashort pulse, whose initial temporal envelope is infinitely smooth with compact support, is considered as it propagates through a temporally dispersive, attenuative medium characterized by two resonance lines. As the propagation distance increases, the accuracy of the popular group velocity description decreases monotonically, whereas the accuracy of the mathematically well-defined asymptotic description and its derivative energy velocity description increase. [S0031-9007(96)02229-6]

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The popular group velocity description of dispersive pulse propagation is widely accepted and employed throughout physics [1] with central importance in optics [2], electromagnetics [3], and acoustics [4]. This description is based on the slowly varying envelope or quasimonochromatic approximation [5–7], originally defined by Born and Wolf [2] in the context of partial coherence. This is a hybrid temporal and frequency domain description [6] in which the temporal field behavior is separated into the product of a slowly varying envelope and an exponential phase term whose angular frequency is centered about some characteristic frequency ω_c. The envelope function is assumed to be slowly varying on the time scale $\Delta t_c \sim 1/\omega_c$, which is equivalent [7] to the assumption that its spectral bandwidth $\Delta\omega$ satisfies the inequality $\Delta\omega/\omega_c \ll 1$ so that the spectral amplitude is sharply peaked about the frequency ω_c.

Recent developments in ultrashort pulse generation techniques have resulted in the production of sub-10-femtosecond optical pulses [8,9]. Because these ultrashort optical pulses do not satisfy the slowly varying envelope approximation, greater care must be taken in modeling their dynamical evolution. The vast majority of researchers still ignore the serious consequences of this simplifying assumption with regard to the effects of linear dispersion in nonlinear optics and related areas. These consequences are presented in this paper through a single representative example from linear optics.

Consider an input pulse envelope modulated sine wave at the plane $z = 0$ that is give by $f(t) = u(t)\sin(\omega_c t)$ with constant applied carrier frequency $\omega_c > 0$ that is propagating in the positive z direction through a linear dielectric whose frequency dispersion is described by the double resonance Lorentz model with complex index of refraction

$$n(\omega) = \left(1 - \frac{b_0^2}{\omega^2 - \omega_0^2 + 2i\delta_0\omega} - \frac{b_2^2}{\omega^2 - \omega_2^2 + 2i\delta_2\omega}\right)^{1/2}. \qquad (1)$$

Here ω_j is the undamped resonance frequency, b_j is the plasma frequency, and δ_j is the phenomenological damping constant of the jth resonance line ($j = 0, 2$). This casual model provides an accurate description of anomalous dispersion in homogeneous, isotropic optical materials when the inequality $\omega_1 \leq \omega_c \leq \omega_2$ is satisfied, in which case the input carrier frequency of the field is contained in the passband between the two absorption bands $[\omega_0, \omega_1]$ and $[\omega_2, \omega_3]$, where $\omega_1 = \sqrt{\omega_0^2 + b_0^2}$ and $\omega_3 = \sqrt{\omega_2^2 + b_2^2}$. The propagated field is given by the exact Fourier-Laplace integral representation [10]

$$A(z,t) = \frac{1}{2\pi}\Re\left\{ i\int_C \tilde{u}(\omega - \omega_c) \times \exp[i(\tilde{k}(\omega)z - \omega t)]d\omega \right\} \qquad (2)$$

for all $z \geq 0$, where $\tilde{u}(\omega)$ is the temporal frequency spectrum of the initial pulse envelope function at the plane $z = 0$. Here $A(z,t)$ represents the scalar wave field whose spectral amplitude $\tilde{A}(z,\omega)$ satisfies the Helmholtz equation $[\nabla^2 + \tilde{k}^2(\omega)]\tilde{A}(z,\omega) = 0$, where $\tilde{k}(\omega) = \beta(\omega) + i\alpha(\omega) = \omega n(\omega)/c$ is the complex wave number of the field with propagation factor $\beta(\omega)$ and attenuation factor $\alpha(\omega)$ in the dispersive, lossy medium. The integral appearing in Eq. (2) is taken over the contour C given by the line $\omega = \omega' + ia$, with a being a fixed constant greater than the abscissa of absolute convergence [10] for the initial envelope function $u(t)$, and where $\omega' = \Re\{\omega\}$ varies from negative to positive infinity.

As a consequence of the quasimonochromatic approximation, the Taylor series expansion of $\tilde{k}(\omega)$ about the carrier frequency ω_c

$$\tilde{k}(\omega) = \sum_{j=0}^{\infty} \frac{1}{j!}\tilde{k}^{(j)}(\omega_c)(\omega - \omega_c)^j, \qquad (3)$$

where $\tilde{k}^{(j)}(\omega) \equiv \partial^j\tilde{k}(\omega)/\partial\omega^j$, may be truncated after only a few terms with some undefined error. Typically, the cubic and higher-order terms in (3) are neglected [1–6] so that one obtains a quadratic dispersion relation. Within this approximation the pulse is found to propagate

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at the classical group velocity $v_g(\omega_c) = [\beta^{(1)}(\omega_c)]^{-1}$, and the quantity $\tilde{k}^{(2)}(\omega_c)$ results in the so-called group velocity dispersion [6].

It is widely believed that the inclusion of just a few additional higher-order terms in the Taylor series approximation of the material dispersion will greatly improve its accuracy [11]. In order to show the fallacy of this assumption, consider the Taylor series expansion of the complex index of refraction (1) for a double resonance Lorentz model of a fluoride glass with an infrared ($\omega_0 = 174.12$ THz, $b_0 = 121.55$ THz, $\delta_0 = 49.55$ THz) and a visible ($\omega_2 = 9144.8$ THz, $b_2 = 6719.8$ THz, $\delta_2 = 1434.1$ THz) resonance line, with associated relaxation times $\tau_{r0} \sim 2\pi/\delta_0 = 126.8f$ sec and $\tau_{r2} \sim 2\pi/\delta_2 = 4.4f$ sec, respectively. The frequency dispersion of the real and imaginary parts of the full double resonance Lorentz model for this glass are depicted by the solid curves in Figs. 1(a), and 1(b), respectively. The Taylor series expansion of $n(\omega) = n_r(\omega) + in_i(\omega)$ is taken about the frequency $\omega_c = 1615$ THz, which occurs at the inflection point in $n_r(\omega)$ where the dispersion is a minimum. The dashed curves in the figure depict the frequency behavior of the four term Taylor series approximation, while the dotted curves in the figure depict the frequency behavior of the ten term Taylor series approximation. Notice that this large increase in the number of terms results in only a slight improvement in the local accuracy of the Taylor series approximation of $n(\omega)$ about ω_c, while the accuracy outside of the passband (ω_1, ω_2) containing ω_c is decreased considerably. The overall accuracy of this approximation decreases as ω_c is moved into either absorption band. Similar results hold for the complex wave number $\tilde{k}(\omega)$. As a consequence:

The inclusion of higher-order terms in the Taylor series approximation of the complex wave number in a dispersive, attenuative medium beyond the quadratic approximation is practically meaningless from both the physical and mathematical points of view.

The inaccuracy of the Taylor series approximation of $\tilde{k}(\omega)$ for ultrawideband pulse propagation in a double resonance Lorentz model dielectric is best illustrated through a specific example. To that end, consider the infinitely smooth, unit amplitude envelope function [12]

$$u(t) = \exp\left\{1 + \frac{\tau^2}{4t(t - \tau)}\right\}, 0 \le t \le \tau \quad (4)$$

and is zero elsewhere, which has compact temporal support with an input full pulse width $\tau > 0$. For a ten cycle pulse at $\omega_c = 1615$ THz, the input full pulse width is $\tau = 20\pi/\omega_c = 38.905f$ sec with equal initial rise and fall times $\tau_R = \tau_F = \tau/2$, where $\Delta\omega/\omega_c \approx 0.25$ with $(\omega_2 - \omega_1)/\omega_c \approx 5.6$ and 99.999% of the input pulse spectral energy is contained in the passband (ω_1, ω_2) between the two resonance lines, as seen in Fig. 1. The dynamical field evolution of this input pulse in the double resonance Lorentz model dielectric, whose frequency dispersion is depicted in Fig. 1, is illustrated

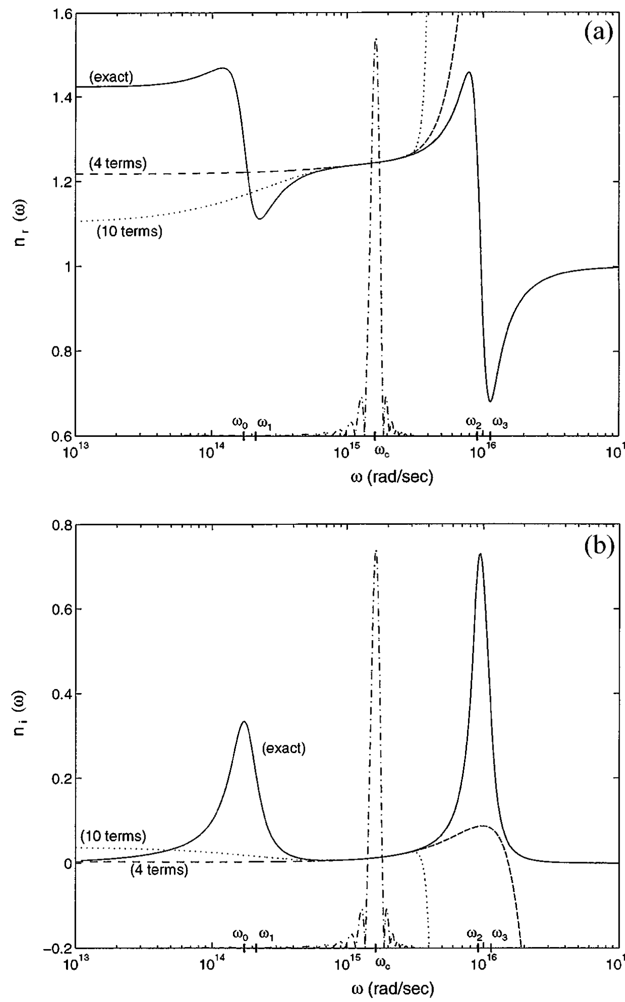

FIG. 1. Frequency dispersion of the (a) real and (b) imaginary parts of the double resonance Lorentz model of the complex index refraction of a fluoride glass with an infrared and a visible resonance line (solid curves). The dashed curves depict the four term and the dotted curves the ten term Taylor series approximations about the minimum dispersion point ω_c between the two resonance lines. For comparison, the relative magnitude (drawn to an arbitrary vertical scale) of the spectrum of the input pulse envelope considered in Fig. 2 is illustrated in both parts by the alternating long and short dashed curves.

in the sequence of diagrams in Fig. 2. Each diagram depicts the temporal field evolution in terms of the dimensionless space-time parameter $\theta = ct/z$ at a fixed propagation distance z relative to the e^{-1} absorption depth $z_d = \alpha^{-1}(\omega_c)$ in the dispersive medium at the input carrier frequency ω_c. The solid curve in each diagram depicts the numerically determined propagated field at that propagation distance using the full Lorentz model (1) of the frequency dispersion, while the dotted curve depicts the numerically determined propagated field using the cubic dispersion relation obtained from the four term Taylor series approximation of $\tilde{k}(\omega)$ about ω_c. The space-time point $\theta_0 = n(0) \cong 1.4238$ marks the value where the peak in the Brillouin precursor will appear [10] in the full dispersion model, while the value $\theta_0' \cong 1.2579$ marks the corresponding space-time value of the cubic dispersion relation. In addition, the space-time point $\theta_{SM} \cong 1.0153$ marks the value when there is a transition from the Sommerfeld precursor to the middle precursor that is characteristic of this double resonance Lorentz model dielectric, and the space-time point $\theta_{MB} \cong 1.2949$ marks the transition from this middle precursor to the Brillouin precursor, as is fully described in the modern asymptotic theory [10,13].

The dynamical field evolution depicted in Fig. 2(a) shows that the exact and approximate descriptions are practically identical at three absorption depths ($z/z_d = 3$) into the dispersive medium, the only observable deviation appearing at both the leading and trailing edges of the pulse. At five absorption depths into the medium ($z/z_d = 5$), this observed deviation at the leading and trailing edges of the pulse has increased significantly, as seen in Fig. 2(b), while the slowly varying envelope approximation remains accurate in its description of the main body of the pulse that is oscillating at (or very near to) the input carrier frequency ω_c. This observed behavior continues as the propagation distance is increased to seven absorption depths ($z/z_d = 7$) in Fig. 2(c) and then to ten absorption depths ($z/z_d = 10$) in Fig. 2(d), at which point the accuracy of the quasimonochromatic approximation is seen to have completely broken down. Similar results are obtained for other input ultrashort pulse widths.

The exact dynamical field evolution depicted by the solid curves in Fig. 2 is accurately provided by the uniform asymptotic description [10] and its derivative energy velocity description [10,14,15] for all $z/z_d \geq 1$, the accuracy increasing as z increases above z_d in the sense of Poincaré [16]. The impulse response of this double resonance Lorentz medium is comprised of a Sommerfeld precursor that dominates the initial field evolution over the space-time domain $1 \leq \theta \leq \theta_{SM}$, whose instantaneous oscillation frequency $\omega_S(\theta) \cong \Re\{\omega_{SP_D^+}(\theta)\}$ chirps down from infinity at $\theta = 1$ and approaches ω_3 from above as $\theta \to \infty$. This is followed by the middle precursor, which dominates the field evolution over the space-time interval $\theta_{SM} \leq \theta \leq \theta_{MB}$, whose instantaneous oscillation frequency $\omega_M(\theta) \cong \Re\{\omega_{SP_M^+}(\theta)\}$ varies over the frequency domain $[0, \omega_2]$ below the upper absorption band. The

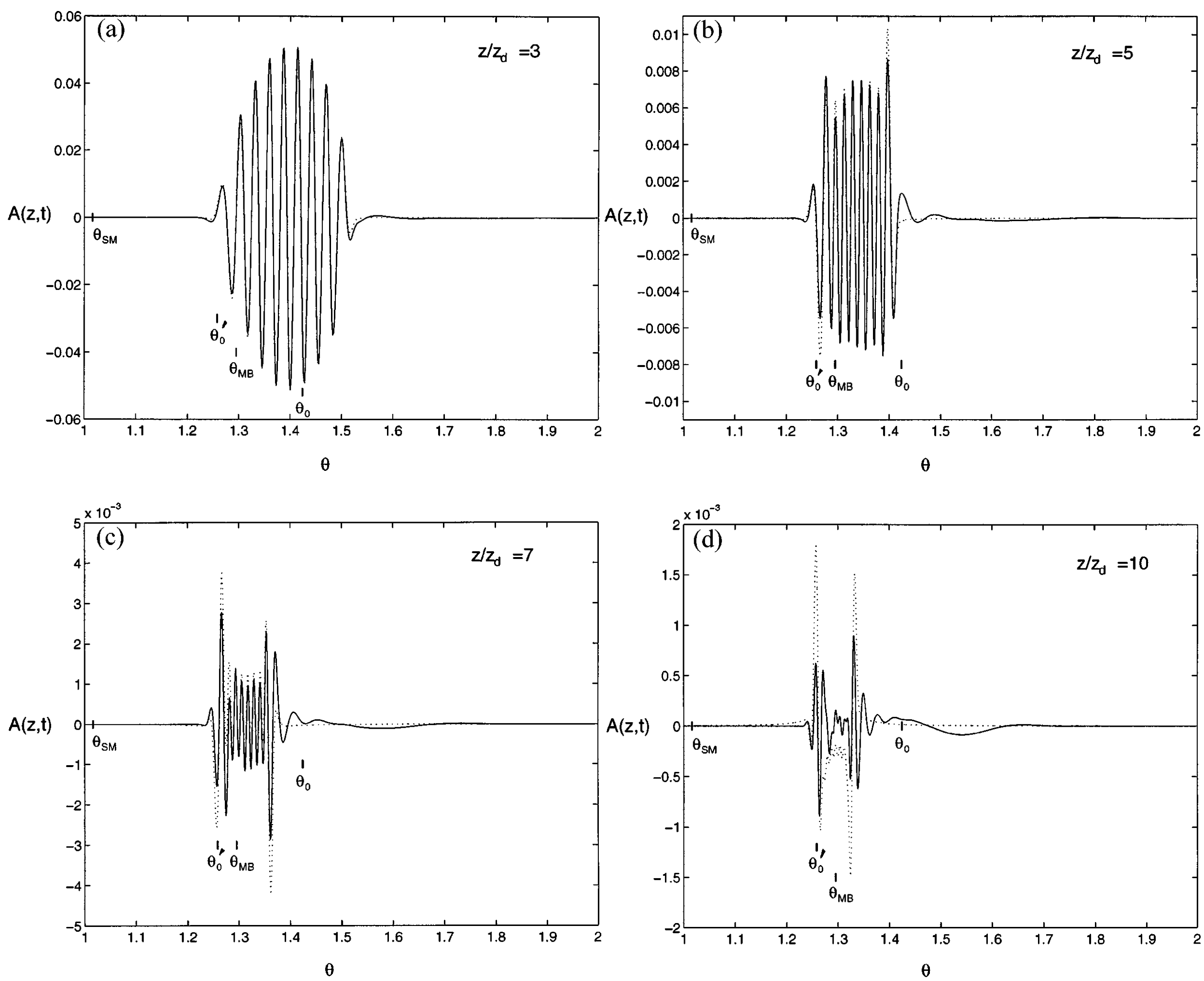

FIG. 2. Numerically determined propagated field evolution using the exact dispersion model (solid curves) and the cubic dispersion approximation (dotted curves) as a function of the dimensionless time parameter $\theta = ct/z$ at several fixed propagation distances z into the dispersive, lossy medium.

Brillouin precursor then dominates the field evolution for all $\theta > \theta_{\mathrm{MB}}$, during which its instantaneous oscillation frequency $\omega_B(\theta) \cong \Re\{\omega_{\mathrm{SP}_M^+}(\theta)\}$ chirps up from its initially quasistatic value and approaches ω_0 from below as $\theta \to \infty$. Here $\omega_{\mathrm{SP}_D^+}(\theta)$ denotes the distant, $\omega_{\mathrm{SP}_M^+}(\theta)$, denotes the middle, and $\omega_{\mathrm{SP}_N^+}(\theta)$ denotes the near saddle point location of the complex phase function $\phi(\omega,\theta) = i(c/z)[\tilde{k}(\omega)z - \omega t] = i\omega[n(\omega) - \theta]$ in the right half of the complex ω plane [10,13]. It then follows that:

As a pulse propagates away from its input plane, its dynamical evolution is initially characterized by the group velocity description, but as the propagation distance increases and the pulse dispersion becomes mature [10,14], *its dynamical evolution becomes characterized by the asymptotic description and its resultant energy velocity description.*

Because $\tau_{R,F} > \tau_{r2}$ in the example considered here, so that the visible resonance relaxation time is dominated by the initial pulse rise/fall time, and since the initial pulse spectrum contains negligible spectral energy above ω_3, the Sommerfeld precursor is absent from the dynamical field evolution depicted in Fig. 2. On the other hand, since $\tau_{R,F} < \tau_{r0}$, so that the infrared resonance relaxation time dominates the initial pulse rise/fall time, the Brillouin precursor will be present in the dynamical field evolution [10]. The front of the pulse is then dominated by the middle precursor and the signal contribution at the input carrier frequency $\omega_c \in [\omega_1, \omega_2]$, which arrives during the evolution of the middle precursor associated with the leading edge of the pulse. As the propagation distance increases above seven absorption depths and the signal contribution at ω_c becomes negligible in comparison to the precursor fields, as in Fig. 2(d), the dynamical field evolution is seen to be comprised of two interfering sets of precursors, one set associated with the leading edge and the other with the trailing edge of the pulse, as described by the uniform asymptotic theory [10]. The propagated pulse then arrives with the evolution of the leading edge middle precursor, followed by the signal contribution oscillating at (or very near to) ω_c and then the trailing edge middle precursor, which is then followed by the interfering leading and trailing edge Brilloiun precursors, as is clearly evident in Fig. 2(d) at $z/z_d = 10$. The peak in the leading edge Brilloiun precursor, which decays only as $z^{-1/2}$, occurs at $\theta = \theta = n(0)$, as properly described by both the asymptotic theory [10] and the energy velocity description [14,15]. Notice, however, that the quasimonochromatic approximation predicts erroneous leading and trailing edge Brillouin-like precursors in place of the middle precursors, the peak in the leading edge Brillouin-like precursor occurring at $\theta = \theta_0'$, as seen in Figs. 2(b)–2(d). At ten absorption depths the quasimonochromatic approximation predicts a peak field amplitude that is approximately twice the actual peak amplitude, and at 15 absorption depths it is approximately 4 times the actual peak amplitude. The implication of this error in the modeling of nonlinear effects in dispersive media may have far-reaching consequences. As a consequence:

The slowly varying envelope or quasimonochromatic approximation of linear dispersive pulse propagation in a double resonance Lorentz model dielectric with $\omega_1 < \omega_c < \omega_2$ *is valid provided that each of the inequalities (in decreasing order of their importance)* $\tau_{R,F} > \max\{2\pi/\delta_0, 2\pi/\delta_2\}$, $\Delta\omega \ll (\omega_2 - \omega_1)$, *and* $\Delta\omega/\omega_c \ll 1$, *are strictly satisfied, where* $\Delta\omega$ *is the spectral width of the input pulse and* $\tau_{R,F}$ *is taken as the smaller of the initial pulse rise and fall times. For either an ultrashort or an ultrawideband pulse with either an initial rise or fall time* $\tau_{R,F}$ *satisfying the inequality* $\tau_{R,F} \lesssim \max\{2\pi/\delta_0, 2\pi/\delta_2\}$,*the accuracy of the quasimonochromatic approximation decreases monotonically as the propagation distance exceeds one absorption depth* $z_d = \alpha^{-1}(\omega_c)$ *at the input carrier frequency* ω_c, *while the accuracy of the asymptotic description increases monotonically.*

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