Inseparable Two Spin-$\frac{1}{2}$ Density Matrices Can Be Distilled to a Singlet Form

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A quantum system is called inseparable if its density matrix cannot be written as a mixture of product states. In this Letter we apply the separability criterion, local filtering, and Bennett *et al.* distillation protocol [Phys. Rev. Lett. **76**, 722 (1996)] to show that *any* inseparable 2×2 system represents the entanglement which, however small, can be distilled to a singlet form. [S0031-9007(96)02199-0]

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After over sixty years, quantum inseparability [1,2] still remains a fascinating object from both a theoretical and experimental point of view. It involves the existence of the entangled pure states, which cannot be written as products of states and which produce a number of nonclassical phenomena. Recently, different applications of the entangled states were proposed, including quantum communication [3], cryptography [4], and quantum computation [5]. In the laboratory, however, one deals with mixed states rather than pure ones. This is due to the uncontrolled interaction with the environment. It involves a fundamental problem of inseparability of quantum system being in a mixed state [6–16]. The system is called separable (inseparable) [17] if its density matrix can be (cannot be) written as a mixture of product states

$$\varrho = \sum_{i=1}^{k} p_i \varrho_i \otimes \tilde{\varrho}_i , \qquad (1)$$

where ϱ_i and $\tilde{\varrho}_i$ are states of the subsystems and $\sum_{i=1}^{k} p_i = 1$. The inseparable states have attracted much attention recently [6–16] as they constitute a natural generalization of pure entangled states. In particular, it has been pointed out [7] that if the system is in an inseparable state then there is no way to ascribe to the subsystems, even in principle, their state vectors.

The fact that in the laboratory we deal with mixed states is a source of a fundamental problem of error correction [11] in quantum computation and quantum communication theory. Within a recently discovered method of transmission of quantum information via inseparable states (teleportation) [3], the problem can be overcome indirectly by the distillation of an ensemble of pairs of particles used subsequently for asymptotically faithful teleportation [10]. Namely, Bennett *et al.* (BBPSSW) [10] considered a protocol which allows one to obtain asymptotically a nonzero number of pairs of spin-$\frac{1}{2}$ particles in the singlet state from a large ensemble described by a density matrix, provided that the latter has fidelity greater than $1/2$. The fidelity of a density matrix ϱ is defined as [11]

$$f = \max \langle \psi | \varrho | \psi \rangle , \qquad (2)$$

where the maximum is taken over all maximally entangled ψ's. The crux of the method is the employment of only local operations and classical communication between "Alice" and "Bob" who share the particles to be distilled [18]. The BBPSSW distillation protocol consists of performing bilateral unitary transformations and measurements over some number of pairs of particles. A similar protocol was used by Peres [15] in collective tests for nonlocality and by Deutsch *et al.* [16] in the context of the security problem in quantum cryptography.

A way of obtaining more entangled mixed states by using local operations and classical communication has been proposed by Gisin [13]. A similar method was used for concentrating of entanglement for pure states by Bennett *et al.* [10]. In Gisin's approach, Alice and Bob subject the particles to the action of local filters, and are able to obtain a mixture which violates Bell's inequality, despite the fact that the original state satisfied them.

Note that the BBPSSW protocol cannot be applied to *all* inseparable states. Indeed, there are states with $f \leq 1/2$ which have nonzero entanglement of formation [11] (hence, cannot be written as a mixture of product states). On the other hand, the filtering method cannot be, in general, applied for the direct production of singlets. However, intuitively one feels that it should be possible to distill an arbitrary inseparable state. It involves a subtle problem of nonlocality of inseparable mixed states. Werner first constructed [6] a family of inseparable mixed states which, nevertheless, admit the local hidden variable (LHV) model for a single von Neumann measurement. Popescu pointed out [8] that some of Werner 2×2 (two spin-$\frac{1}{2}$) states admitting the LHV model are useful for quantum teleportation, and he showed [9] that most of Werner mixtures reveal nonlocality, if one takes into account the sequences of

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measurements. Then he conjectured that all inseparable states are nonlocal. This question could be solved just by showing that each inseparable state can be distilled. Indeed, the distillation process can be considered as a sequence of measurements performed on the collections of pairs of particles, and, as the obtained pairs in singlet state violate local realism, then the original ensemble also does [9,12]. The problem is that we do not have complete "operational" characterization of the inseparable mixed states. Fortunately, quite recently, an effective criterion of separability of mixed states for 2×2 and 2×3 systems has been found [14]. Here, using the criterion, filtering, and BBPSSW protocol, we will show that *any* inseparable mixed two spin-$\frac{1}{2}$ state can be distilled to obtain asymptotically faithful teleportation. In particular, as we shall see, if one replaces filtering by generalized measurements (to avoid losing particles) higher efficiency of distillation can be obtained by means of a recursive process.

It has been shown [14] that a state ϱ of a 2×2 system is separable if, and only if, its partial transposition ϱ^{T_2} is a non-negative operator, i.e., if all eigenvalues ϱ^{T_2} are non-negative. Here the partial transposition ϱ^{T_2} associated with the arbitrary product orthonormal $f_i \otimes f_j$ basis is defined by the matrix elements in this basis:

$$\varrho^{T_2}_{m\mu,n\nu} \equiv \langle f_m \otimes f_\mu | \varrho^{T_2} | f_n \otimes f_\nu \rangle = \varrho_{m\nu,n\mu} \,. \quad (3)$$

Clearly the matrix ϱ^{T_2} depends on the basis, but its eigenvalues do not. Hence one can check separability using an arbitrary product orthonormal basis in Hilbert space $C^2 \otimes C^2$.

Suppose now that ϱ is inseparable, and let ψ be an eigenvector associated with some negative eigenvalue of ϱ^{T_2}. Since in the process of distillation Alice and Bob can perform local unitary transformations, we can assume without loss of generality that ψ is of the form

$$\psi = a e_1 \otimes e_1 + b e_2 \otimes e_2 \,, \quad (4)$$

where $\{e_i\}$ form the standard basis in C^2 and $a, b \geq 0$. Now $\langle \psi | \varrho^{T_2} | \psi \rangle < 0$ implies

$$\langle I \otimes W \psi_2 | \varrho^{T_2} | I \otimes W \psi_2 \rangle < 0 \,, \quad (5)$$

where $\psi_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2)$ and

$$W = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}. \quad (6)$$

Let us denote by $\tilde{\varrho}$ the state emerging after performing the operation given by $I \otimes W$,

$$\tilde{\varrho} = \frac{I \otimes W \varrho I \otimes W}{\mathrm{Tr}\, I \otimes W \varrho I \otimes W} \,. \quad (7)$$

This state describes the subensemble of the pairs of particles, which passed the local filter described by the operator W. Now the inequality (5) implies

$$\mathrm{Tr}\, P_2^{T_2} \tilde{\varrho} < 0 \,, \quad (8)$$

where $P_2 = |\psi_2\rangle\langle\psi_2|$. Note that $P_2^{T_2}$ (up to a positive factor) is equal to the operator V given by $V \phi \otimes \tilde{\phi} = \tilde{\phi} \otimes \phi$, which was used by Werner [6] in the necessary condition $\mathrm{Tr}\, \varrho V \geq 0$ for separability. Thus Eq. (8) is equivalent to

$$\mathrm{Tr}\, V \tilde{\varrho} < 0 \,. \quad (9)$$

Now, based on the results of [7], it is easy to show that the above inequality implies [19]

$$\mathrm{Tr}\, P_0 \tilde{\varrho} > \frac{1}{2} \,, \quad (10)$$

where P_0 denotes the singlet state. The last relation shows us that the state $\tilde{\varrho}$ can be distilled by the BBPSSW protocol.

To summarize, given sufficiently many pairs of particles in an inseparable state, Alice and Bob can distill from it a nonzero number of singlets. To this end, they first perform a measurement by means of a complete set of product observables on some number of particles to get the matrix elements of the state describing the ensemble (it still involves only local operations and classical communication). Then they perform suitable product unitary transformations. Subsequently, Alice directs her particles toward a filter, the parameters of which can be derived from the density matrix describing the ensemble. Then Alice informs Bob as to which particles have not been absorbed by the filter so that he can discard the particles which lost their counterparts. The subensemble obtained in this way can now be subjected to the BBPSSW protocol to distill singlets. If the efficiency (the number of distilled pairs divided by the number of noisy pairs) of the latter protocol is given by η, then the efficiency ε of the whole process is given by

$$\varepsilon = \eta p \,, \quad (11)$$

where $p = \mathrm{Tr}(I \otimes W \varrho I \otimes W)$ is the probability of passing the filter, i.e., the efficiency is the product of the efficiencies of two stages: filtering and BBPSSW protocol.

Although the distillation protocol described above is effective in the sense that given any inseparable state, one can always distill a nonzero number of singlets, it does not have to be the best possible one. It seems that for the inseparable states with $f \leq 1/2$ the best possible protocol should certainly consist of filtering as the first stage; nevertheless, better efficiency of this stage can be obtained. Consider, for example, the family of states introduced in the context of inseparability and Bell inequalities [20]

$$\varrho = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2| \,, \quad (12)$$

where $|\psi_1\rangle = c e_1 \otimes e_1 + d e_2 \otimes e_2$, $|\psi_2\rangle = c e_1 \otimes e_2 + d e_2 \otimes e_1$ where $c, d > 0, p \neq 1/2$, and $\{e_i\}$ form the standard basis in C^2. All the above states are inseparable. Here, one should not follow the protocol described above, but rather to apply the operation $W \otimes I$ with

$$W = \begin{bmatrix} d & 0 \\ 0 & c \end{bmatrix}. \quad (13)$$

The efficiency of the first stage can also be raised by replacing the filter with the generalized measurement, one of the outcomes of which would produce the same result as filtering. The generalized measurement is given by a partition of unity $\{V_i\}$, where $\sum V_i V_i^\dagger = I$. After the ith outcome is obtained (provided nondegeneracy of the measurement), the state ϱ collapses into

$$\varrho_i = \frac{V_i \varrho V_i^\dagger}{\mathrm{Tr}(V_i \varrho V_i^\dagger)}. \tag{14}$$

Thus instead of a filter, one can use a generalized measurement, and choose the particles which produced suitable outcome k. The advantage here is that, if some other outcome was obtained, the particle is not lost as in the case of filtering. It may be the case that the ensemble of the particles which did not produce the required outcome would still be described by some inseparable density matrix. Then one can repeat the procedure, changing suitably the partition of unity, to distill the subensemble. In this way we obtain a recursive process, the efficiency of which is higher than in the case of single filtering.

Now we will discuss our distillation protocol by means of a geometrical representation of the state [7]. For this purpose note that an arbitrary two spin-$\frac{1}{2}$ state can be represented in the Hilbert-Schmidt (H-S) space of all operators acting on $C^2 \otimes C^2$ as follows:

$$\varrho = \frac{1}{4}\Big(I \otimes I + \boldsymbol{r} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \boldsymbol{s} \cdot \boldsymbol{\sigma} + \sum_{m,n=1}^{3} t_{nm} \sigma_n \otimes \sigma_m\Big). \tag{15}$$

Here I stands for identity operator, $\boldsymbol{r}$, $\boldsymbol{s}$ belong to R^3, $\{\sigma_n\}_{n=1}^3$ are the standard Pauli matrices, and $\boldsymbol{r} \cdot \boldsymbol{\sigma} = \sum_{i=1}^3 r_i \sigma_i$. The coefficients $t_{mn} = \mathrm{Tr}(\rho \sigma_n \otimes \sigma_m)$ form a real matrix denoted by T. The vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ describe local properties of the state while the T matrix describes a kind of projection of ϱ onto the set of states generated by maximally entangled projectors. (See Ref. [7] and references therein for more details concerning the formalism of the H-S space of the 2×2 system.) Thus the T matrix determines whether the state can be directly subjected to BBPSSW protocol to produce nonzero asymptotic singlets. Indeed, based on the results of Ref. [7], one obtains that $f > 1/2$ if, and only if, $N(\varrho) > 1$, where $N(\varrho) = \mathrm{Tr}\,\sqrt{T^\dagger T}$, and then

$$f = \frac{1}{4}[1 + N(\varrho)]. \tag{16}$$

For example, many of the states (12) have $N(\varrho) \leq 1$, hence they *cannot* be distilled by the BBPSSW protocol itself. To find the Bell operator basis [21] in which a given state has the highest fraction of a maximally entangled vector, it suffices to find rotations which diagonalize the T matrix. Subsequently, using the homomorphism between the group unitary transformations of two level systems and rotation group [22], one can find the suitable product unitary transformation which will convert the standard Bell basis into the best one for the considered state.

Further, we will assume that T is diagonal so that it can be treated as a vector in R^3. It has been proven [7] that if ϱ is a state then T must belong to the tetrahedron $\mathcal{T}$ with vertices $(-1,-1,-1)$, $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$ (see, in this context, [11]). Again, if ϱ is separable then T must belong to the octahedron $\mathcal{L}$ which is a cross section of $\mathcal{T}$ and $-\mathcal{T}$ (see Fig. 1).

For the states with $\boldsymbol{r} = \boldsymbol{s} = 0$ (we call them T states), the above conditions are also sufficient [7], hence the set of T states is equal to the tetrahedron $\mathcal{T}$ and the set of separable T states can be identified with the octahedron $\mathcal{L}$ (note that $\mathcal{L}$ is described by inequality $N(\varrho) \leq 1$ [7]).

Consider now the following case, when the T matrix of a given state lies outside the octahedron (we will say that the state lies outside the octahedron). Then, according to [7], there exists some maximally entangled state ψ such that $|\langle\psi|\varrho|\psi\rangle| > 1/2$. Thus, the state can be distilled by the BBPSSW protocol. Suppose now that the state lies inside the octahedron. Then the first step of the BBPSSW protocol (random bilateral unitary transformations) will destroy any inseparability of the state. Indeed, there are two consequences of this step. The first one is that local parameters become $\boldsymbol{r} = \boldsymbol{s} = 0$ (as a consequence of random rotations of vectors $\boldsymbol{r}, \boldsymbol{s}$ inside a Bloch sphere). The second, very important, one is that, after the randomizing procedure, the T matrix still remains inside the octahedron (taking into account remarks from the previous paragraph, it is easy to see that otherwise one could produce inseparable T states from separable T states by use of local operations, which is obviously impossible).

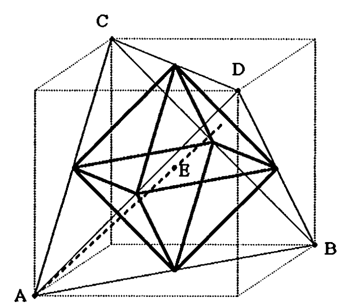

FIG. 1. For the states with diagonal T matrix the latter can be treated as a vector in R^3. In particular, the projectors $\{P_i\}$ corresponding to the Bell operator basis are uniquely represented by the points $A = (-1,-1,-1)$, $B = (1,1,-1)$, $C = (1,-1,1)$, and $D = (-1,1,1)$. Then (i) for any state, its T matrix must belong to the tetrahedron $ABCD$ via the condition $\mathrm{Tr}\,\varrho P_i \geq 0$; (ii) for a separable state, T must belong to the bold-line-contoured octahedron, by virtue of the additional condition $\mathrm{Tr}\,\varrho P_i^{T_2} \geq 0$. Random bilateral unitary transformations "project" the T matrix onto the dashed line. For a state with $\mathrm{Tr}\varrho P_0 > 1/2$, the outputs of the subsequent iterations of the BBPSSW protocol will lie on the line, closer and closer to the singlet state represented by the point A.

Thus, according to the characterization of T states, the output state will be separable.

Now, the role of filtering becomes clear. Namely, this procedure allows one to transfer the entanglement hidden in the relations between $\boldsymbol{r}$, $\boldsymbol{s}$, and T to the T matrix itself. If the input state is inseparable, but still lies inside the octahedron, the process of filtering will move it outside, so that the BBPSSW protocol will produce a nonzero number of singlets.

In conclusion, we have shown that any inseparable mixed state of a two spin-$\frac{1}{2}$ system represents the entanglement which, however small, can be distilled to a singlet form by using local operations and an exchange of classical information. It follows, in the context of the LHV models for sequential measurements [12], that, if considered collectively, any inseparable 2×2 system reveals nonlocal properties. Clearly it solves the problem of nonlocality of inseparable mixtures [8,9] for 2×2 systems. Finally, it is interesting to note that distillability of an arbitrary inseparable mixed state of a 2×2 system is exactly connected with the negative eigenvalue of partial transposition of the state. Thus the possibility of distillation may be here interpreted as a nonlocal effect "produced" by the eigenvalue.

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