Solutions to the Time Dependent Schrödinger and the Kadomtsev-Petviashvili Equations

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A method to obtain a new class of discrete eigenfunctions and associated real, nonsingular, decaying, "reflectionless" potentials to the time dependent Schrödinger equation is presented. Using the inverse scattering transform, related solutions of the Kadomtsev-Petviashvili equation are found. The eigenfunctions have poles of order m, $m > 1$ in the complex plane and are also characterized by an index, or "charge," which is obtained as a constraint in the theory. [S0031-9007(96)02189-8]

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In this Letter we describe a method to obtain a new class of decaying potentials and corresponding solutions to the time dependent Schrödinger and Kadomtsev-Petviashvili-I (KPI) equations. Both of these equations have wide application in physics. The Schrödinger operator is centrally important in quantum mechanics and the KP equation is a ubiquitous nonlinear wave equation governing weakly nonlinear long waves in two dimensions with slow transverse variations. It is well known that the nonstationary Schrödinger operator can be used to linearize the KPI equation via the inverse scattering transform [IST; (see e.g., [1])]. In [2] it was shown by IST that certain discrete states associated with complex conjugate pairs of simple eigenvalues of the Schrödinger operator are related to lump type soliton solutions which decay as $O(1/r^2)$, $r^2 = x^2 + y^2$. The lump solutions of KPI have been extensively studied since they were first found by direct methods [3].

Here we demonstrate that there are real, nonsingular, decaying potentials of the Schrödinger operator which correspond to discrete states with multiple eigenvalues. Corresponding to real potentials, one class of these eigenfunctions has poles/eigenvalues in the upper (lower) half plane of multiplicity, order m ($m > 1$), but in the lower (upper) half plane the conjugate eigenvalue is simple. We give explicit formulas for $m = 2, 3$. These solutions decay as $O(1/r^2)$. Thus, unlike the time independent Schrödinger equation, there are real, decaying potentials of the time dependent Schrödinger equation which are related to eigenvalues with multiplicity. We note, however, that these potentials are not absolutely integrable, and this underlies the fact that the potentials are characterized by the order of the pole and a topological number, an index or winding number, which we refer to as the charge.

The properties of the multilump solutions are interesting. They can be thought of as a collection of m individual humps which interact in a nontrivial manner. They have m maxima which move with different asymptotic velocities as $t \to \pm\infty$.

In what follows we outline the methods and give the main results. The nonstationary Schrödinger equation is taken in the form

$$i\psi_y + \psi_{xx} + u\psi = 0\,. \tag{1}$$

We use y as the "time" variable in Eq. (1) because, via IST, y plays the role of the transverse coordinate in the solution $u(x, y, t)$ of the KPI equation:

$$(u_t + 6uu_x + u_{xxx})_x = 3u_{yy}\,. \tag{2}$$

Equation (2) is obtained from the compatibility of (1) and

$$\psi_t + 4\psi_{xxx} + 6u\psi_x + 3u_x\psi - 3i\left(\int_{-\infty}^{x} u_y\,dx'\right)\psi + \alpha\psi = 0\,, \tag{3}$$

where α is an arbitrary constant.

The discrete states we are concerned with here are the eigenvalues of the Fredholm integral equation,

$$\mu(x,y;k) = 1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x - x', y - y'; k) \times (u\mu)(x', y'; k)\,dx'\,dy' \tag{4}$$

(hereafter the limits on all double integrals are from $-\infty, \infty$), where $\psi = \mu \exp[i(kx - k^2 y)]$, $\mu(x,y;k) \equiv \mu(k)$, and $G(x,y;k) = [1/(2\pi)^2]\int\int \hat{G}(p,k)\exp(ipx + iqy)\,dp\,dq$, with $\hat{G}(p,q;k) = 1/(p^2 + 2pk + q)$. For real k there are two limits: $G_\pm(k)$, where $\pm$ denotes the limit from $\mathrm{Im}k > 0$ and $\mathrm{Im}k < 0$, respectively; $G_\pm(k)$ are analytic in their respective half planes (cf. [1,2]). From Eq. (4) and $G_\pm(k)$ one constructs two eigenfunctions $\mu_\pm(k)$ which are meromorphic in the upper/lower half k-planes. These eigenfunctions lead to a nonlocal Riemann-Hilbert problem from which the inverse scattering is carried out. For ease of discussion, here we will concern ourselves with pure pole solutions of μ, where $\mu_+ = \mu_-$. Continuous spectrum can be added, but we shall not do so here.

The usual discrete states/lump solutions can be found by solving the following linear algebraic system of equations (see, e.g., [1,2]),

$$1 + \sum_{\substack{m=1\\ m\neq j}}^{2n} \frac{\phi_m}{k_j - k_m} = -if_j\phi_j, \qquad j = 1,\dots 2n\,, \tag{5}$$

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where $f_j = f(k_j) = x - 2k_j y + \gamma_j(t)$ and the eigenfunction $\mu(k)$ of (4) is written

$$\mu(k) = 1 + \sum_{m=1}^{2n} \frac{\phi_m}{(k - k_m)}\,. \qquad (6a)$$

The corresponding "reflectionless" potentials are given by

$$u = -2i \frac{\partial}{\partial x} \sum_{m=1}^{2n} \phi_m = 2 \frac{\partial^2}{\partial x^2} \ln F\,, \qquad (6b)$$

where F is the determinant of the coefficient matrix of the system (5). Real, nonsingular potentials are obtained when $k_{j+n} = \overline{k}_j$ and $\gamma_{j+n}(t) = \overline{\gamma}_j$, $j = 1, \ldots, n$. Lumps for KPI are obtained when one inserts the proper time dependence for $\gamma_j(t)$, obtained from (3). It is found that $\gamma_j(t) = 12k_j^2 t + \gamma_{j0}$, where $\gamma_{j,0} \equiv \gamma_j(t = 0)$. We now describe for the case $n = 2$ in the above formulas (two pairs of conjugate eigenvalues in the upper/lower half planes) how, by appropriately coalescing the simple eigenvalues of the Schrödinger operator, we can find discrete states which correspond to eigenvalues of multiplicity 2 in (say) the upper half plane and simple eigenvalues in the lower half plane. In the limiting process we take $k_1 = k_2 + \varepsilon$, $k_3 = \overline{k}_1$, $k_4 = \overline{k}_2$, $k_1 = a + ib$, $|\varepsilon| \ll 1$, expand as follows:

$$\phi_m = \frac{\phi_m^{(-1)}}{\varepsilon} + \phi_m^{(0)} + \phi_m^{(1)}\varepsilon + \ldots\,,$$

$$\gamma_{m,0} = \gamma_{m,0}^{(-1)}/\varepsilon + \gamma_{m,0}^{(0)} + \gamma_{m,0}^{(1)}\varepsilon + \ldots\,, \quad m = 1, \ldots 4\,,$$

and substitute these expansions into Eq. (5). At order $1/\varepsilon^2$ there are various possibilities. But reality forces a condition on $\gamma_{m,0}^{(-1)}$. We restrict to $\gamma_{1,0}^{(-1)} = \gamma_{4,0}^{(-1)} = -\gamma_{2,0}^{(-1)} = -\gamma_{3,0}^{(-1)} = -i$. In order for μ to have a finite limit as ε tends to zero, we are forced to take $\phi_1^{(-1)} + \phi_2^{(-1)} = \phi_3^{(-1)} + \phi_4^{(-1)} = 0$. Proceeding to subsequent orders is straightforward. It is convenient to define new variables $\phi_1^{(0)} + \phi_2^{(0)} \equiv \Phi_1$, $\phi_3^{(0)} + \phi_4^{(0)} = \Phi_{\overline{1}}$, $\phi_1^{(-1)} \equiv \Psi_2$. We find that $\phi_3^{(-1)} = \phi_4^{(-1)} = 0$, $\phi_1^{(-1)} = i\Phi_1/(f + \gamma^{(0)})$, $\gamma^{(0)} \equiv \gamma_{1,0}^{(0)}$, $f \equiv f(k_1)$, $\phi_3^{(0)} = \phi_4^{(0)}$, and in the limit $\varepsilon \to 0$ the eigenfunction μ then has the following spectral structure:

$$\mu = 1 + \frac{\Psi_2}{(k - k_1)^2} + \frac{\Phi_1}{(k - k_1)} + \frac{\Phi_{\overline{1}}}{(k - \overline{k}_1)}\,. \qquad (7)$$

The following system of equations is obtained:

$$\frac{g}{2if}\Phi_1 - \alpha\Phi_{\overline{1}} = 1\,, \qquad (8a)$$

$$\left[\frac{2\overline{\alpha}^3}{f\overline{f}} + \frac{\overline{\alpha}^2}{i}\left(\frac{1}{f} + \frac{1}{\overline{f}}\right) - \overline{\alpha}\right]\Phi_1 + \frac{\overline{g}}{2i\overline{f}}\Phi_{\overline{1}} = 1\,, \qquad (8b)$$

where

$$\alpha = \frac{1}{k_1 - \overline{k}_1}, \qquad g = g(k_1) = f^2 + \delta(t) - 2iy\,,$$

$$\delta(t) = \delta^{(0)} + 24ik_1 t, \qquad (\delta^{(0)} = \gamma_{1,0}^{(1)} - \gamma_{2,0}^{(1)})\,.$$

Finally, the solution, a reflectionless potential or a multipole lump order-2 to the KPI equation, is obtained from (6b),

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln F_2 \qquad (9a)$$

$$F_2 = (z'^2 - 4b^2 y'^2 - 24bt + \delta_R)^2 + \left[2y'(1 + 2bz') + \frac{\gamma_I}{b} - \delta_I\right]^2 + \frac{1}{b^2}\left[\left(z' - \frac{1}{2b}\right)^2 + 4b^2 y'^2 + \frac{1}{4b^2}\right], \qquad (9b)$$

where $k_1 = a + ib$, $z' = x' - 2ay'$, $x' = x - 12(a^2 + b^2)t - x_0$, $y' = y - 12at - y_0$, $x_0 = (\gamma_I a - \gamma_R b)/b$, $y_0 = \gamma_I/2b$.

Actually this particular solution of KPI was constructed some time ago [4], but without reference to the underlying scattering problem. It was studied more recently in [5], again purely in terms of KP solutions and related dynamics. The present approach serves to put the class of multipole lump solutions into a unified framework via scattering theory of the time dependent Schrödinger operator.

This multipole order-2 configuration for the KPI equation is composed of two mutually interacting humps, each of which moves with distinct asymptotic velocities. For large t the maxima of the humps are located approximately at the zeros of F. Both humps have a maximum amplitude $u(x_\pm, y_\pm) \sim 16b^2$ as $t \to \pm\infty$. Assuming $b > 0$, as $t \to -\infty$ the two maxima (+ denotes fast, − denotes slower hump) are located at

$$x_\pm \sim 12(a^2 + b^2)t \pm \sqrt{24a^2|t|/b} + x_0 + 2ay_0 - 1/2b\,,$$

$$y_\pm \sim 12at \pm \sqrt{6|t|/b} + y_0\,,$$

and as $t \to +\infty$, $x_\pm \sim 12(a^2 + b^2)t \pm \sqrt{24bt} + x_0 + 2ay_0$, $y_\pm \sim 12at + y_0$.

These results clearly show that the humps display a nontrivial interaction. As $t \to \pm\infty$ the humps diverge from one another, proportional to $|t|^{1/2}$. This situation is different from the "pure" lump case where each lump has the same velocity as $t \to \pm\infty$.

We next outline how the class of multipole lump solutions can be obtained directly as discrete multiple eigenvalue states of the time dependent Schrödinger equation.

In the case when the eigenfunction $\mu(k)$ consists only of simple poles, substitution of μ given by (6a) into the integral equation (4), noting that the eigenfunctions ϕ_m are homogeneous solutions of the integral equation, and using properties of the Greens function and Fredholm theory, the following equation is found [2]:

$$\nu(k_j) = -if(k_j)\phi_j\,, \qquad (10)$$

where in general we denote $\nu(k_j) \equiv \lim_{k \to k_j}$ (μ-singular part of μ); in this case $\nu(k_j) = \lim_{k \to k_j}[\mu - \phi_j/k - k_j)]$. In addition, the constraint $Q(k_j, \phi_j) = 1$ is required, where

$$Q(k_j, \phi_j) \equiv \frac{i}{2\pi} \operatorname{sgn}(\operatorname{Im} k_j) \int\int u\phi_j \, dx\, dy\,. \qquad (11)$$

We call $Q(k_j, \phi_j)$ the charge or index; we discuss this more fully later. Thus for simple poles, which are related to usual lumps of KPI, $Q(k_j, \phi_j) = 1$.

Other eigenfunctions with higher order poles can be described within this framework. We define the charge in the same way. Then, substitution of the eigenfunction μ, with a double pole in the upper half plane/single in the lower half plane, given by (7), into the integral equation (4) with the coefficients $\Psi_2, \Phi_{\bar{1}}$ being homogeneous solutions, and again using the properties of the Greens function and Fredholm theory yields the following equations:

(a) At $k = k_1$,

$$\Phi_1 = -if(k_1)\Psi_2; \qquad \nu(k_1) = -g\Psi_2/2\,, \qquad (12a)$$

with the constraints $Q(k_1, \Phi_1) = 2$ and $\int\int u\Psi_2 dxdy = 0$;

(b) at $k = \bar{k}_1$,

$$\left[\frac{d\nu}{dk} + if(k)\nu\right]_{k=\bar{k}_1} = \frac{1}{2}\bar{g}\Phi_{\bar{1}}\,, \qquad (12b)$$

with $Q(\bar{k}_1, \Phi_{\bar{1}}) = 2$, where g and the definition of ν are given above,

$$\nu(k_1) = \lim_{k\to k_1}[\mu - \Psi_2/(k-k_1)^2 - \Phi_1/(k-k_1)]\,,$$

$$\nu(\bar{k}_1) = \lim_{k\to \bar{k}_1}[\mu - \Phi_{\bar{1}}/(k-\bar{k}_1)]\,.$$

Inserting the representation for μ, Eq. (7), into Eqs. (12a) and (12b) for $\nu(k)$ yields the system of equations (8) and the reflectionless potential/multipole lump order-2 given by Eq. (9). We say that this solution has charge=2. We see that the quantities Q may take on other values different than 1, which was the case for standard simple pole-lump solutions.

The method extends to higher order multipoles. The main results are the following. The eigenfunction $\mu(k)$ has the expansion

$$\mu(k) = 1 + \frac{\Psi_3}{(k-k_1)^3} + \frac{\Psi_2}{(k-k_1)^2} + \frac{\Phi_1}{k-k_1} + \frac{\Phi_{\bar{1}}}{k-\bar{k}_1}\,. \qquad (13)$$

Substituting (13) into (4) and noting that the functions $\Psi_3, \Phi_{\bar{1}}$ are homogeneous solutions yields the following equations and constraints:

(a) At $k = k_1$,

$$\Psi_2 = 2i(f/g)\Phi_1 = -if\Psi_3, \qquad \nu(k_1) = -(h/3g)\Phi_1$$

with $Q(k_1, \Phi_1) = 3$ and $\int\int u\Psi_j dxdy = 0$; $j = 2, 3$;

(b) at $k = \bar{k}_1$,

$$\left[\frac{d^2\nu}{dk^2} + 2if\frac{d\nu}{dk} - \bar{g}\nu + \frac{1}{3}\bar{h}\Phi_{\bar{1}}\right]_{k=\bar{k}_1} = 0\,,$$

with $Q(\bar{k}_1, \Phi_{\bar{1}}) = 3$. In the above formulas, g was given earlier, and we define $h(k_1) \equiv h \equiv if^3 + [3i\delta(t) + 6y]f + \beta(t)$. From Eq. (6) we obtain the reflectionless potential/multipole lump order-3 of KPI,

$$u = 2\frac{\partial^2}{\partial x^2}\ln F_3$$

$$\begin{aligned} F_3 = {} & [z'^3 - 12b^2y'^2z' + 3z'(\delta_R - 24bt) + 6by'(\delta_I - 2y') + \beta_R - 24t]^2 \\ & + [8b^3y'^3 - 6bz'^2y' + 3z'(\delta_I - 2y') - 6by'(\delta_R - 24bt) + \beta_I]^2 \\ & + \frac{9}{4b^2}\left[\left(z'^2 - 4b^2y'^2 + \delta_R - 24bt - \frac{z'}{b} + \frac{1}{2b^2}\right)^2 + (4y'bz' - \delta_I)^2 + \frac{1}{b^2}\left(z' - \frac{1}{b}\right)^2 + 4y'^2 + \frac{1}{4b^4}\right], \end{aligned} \qquad (14)$$

where we note that the time dependence of the KP solution is obtained from Eq. (3); $\gamma(t)$, $\delta(t)$ is determined to be as given before, and $\beta(t) = 24it + \beta_0$. Since the charges $Q(k_1, \Phi_1) = Q(\bar{k}_1, \Phi_{\bar{1}}) = 3$ we say that this solution has charge=3.

This multipole order-3 configuration for the KPI equation is composed of three mutually interacting humps, which move with distinct asymptotic velocities. For large t the maxima of the humps are located approximately at the zeros of F_3. As $t \to \pm\infty$ the three maxima are located at (assuming $b > 0$) (x'_+, y'_+), (x'_0, y'_0), (x'_-, y'_-), where $x'_0 = 12(a^2 + b^2)t + x_0$, $y'_0 = 12at + y_0$, and as $t \to -\infty$ $x'_\pm \sim x'_0 - 4/3b$, $y'_\pm \sim y'_0 \pm \sqrt{18|t|/b}$, and as $t \to +\infty$, $x'_\pm \sim x'_0 \pm \sqrt{72bt} + 1/(6b)$, $y'_\pm \sim y'_0$.

Higher order multipole lumps $m > 3$ can be constructed in the same manner as outlined here, but we will not dwell on that. Moreover, other classes of solutions can also be obtained via similar techniques. For example, if the eigenfunction μ has the spectral structure

$$\mu = 1 + \frac{\Psi_2}{(k-k_1)^2} + \frac{\Phi_1}{(k-k_1)} + \frac{\Psi_{\bar{2}}}{(k-\bar{k}_1)^2} + \frac{\Phi_{\bar{1}}}{(k-\bar{k}_1)}\,, \qquad (15)$$

with $Q(k_1, \Phi_1) = Q(\bar{k}_1, \Phi_{\bar{1}}) = 3$, then the following solution results, $u = 2\partial^2(\ln G_3)/\partial x^2$, where

$$G_3 = (z'^3 - 12b^2y'^2z' + 12t + \beta_R)^2 + (8b^3y'^3 - 6bz'^2y' + \beta_I)^2 + \frac{9}{4b^2}\left\{\left[\left(z' - \frac{1}{2b}\right)^2 + 4b^2y'^2 + \frac{1}{4b^2}\right]\left[\left(z' + \frac{1}{2b}\right)^2 + 4b^2y'^2 + \frac{1}{4b^2}\right]\right\}, \tag{16}$$

with $\int\int u\Psi_j\,dx\,dy = 0$, $j = 2, \overline{2}$.

These real, nonsingular, reflectionless potentials of the time dependent Schrödinger operator/multipole-lump solutions of KPI are most simply characterized by the order of the pole in the spectral theory and their associated charge, or index. Furthermore, consideration of spectral functions formed by superposition of, say, terms like (7) with (double) poles at $k_1, \dots, k_r$ and single poles at $\overline{k}_1, \dots, \overline{k}_r$ with $Q_1, \dots, Q_r = 2$ implies a $2r \times 2r$ linear system whose solution yields a $2r$ hump interacting solution of KPI. Similar considerations apply to superposition of terms like, say, (13) or (15). One can also consider a mixture of multiple poles with different charges $Q_1, \dots, Q_j$. The equations which define the solution and overall interaction of humps can be found via the above method. In this way a broad class of new real, nonsingular decaying solutions of KPI with interesting physical properties can be obtained.

Finally, we discuss the notion of the charge or index Q. Interestingly, this quantity Q has a topological interpretation. Indeed, for lump solutions the residue Φ of the pole at, say, k_1 has the asymptotic form $\Phi = (\ln\eta)' + O(1/r^2)$, $r \equiv \sqrt{x^2 + y^2}$, where the derivative is taken along a contour Γ_∞ that surrounds the origin once. Then: $Q = \text{index}\ \eta$.

For example, Eq. (10) calling $\phi_j \equiv \Phi$, $k_j \equiv k$, $f_j \equiv f$ implies that $\Phi = \frac{i}{f} + O(\frac{1}{r^2})$. Using $\Phi_{xx} + 2ik\Phi_x + i\Phi_y + u\Phi = 0$ and Green's theorem we have

$$\begin{aligned} Q(k,\Phi) &\equiv \frac{i}{2\pi}\,\mathrm{sgn}(\mathrm{Im}k)\int\int u\Phi\,dx\,dy \\ &= \frac{-1}{2\pi}\int_{\Gamma_\infty}\frac{dx - 2kdy}{f} \\ &= \frac{1}{2\pi i}\int_{\Gamma_\infty}\frac{df}{f} = \frac{1}{2\pi i}\int_{f(\Gamma_\infty)}\frac{dz}{z} = 1\,, \end{aligned} \tag{17}$$

and we see from (17) that Q = winding number of f. The method for the double pole with charge=2 is similar. Substituting the representation (7) into (1) (recalling $\psi = \mu\exp[i(kx - k^2y)]$, integrating over x, y and using Green's theorem and the properties of the solution from (12a) yields, in the same manner as (17),

$$Q(k_1,\Phi_1) = \frac{1}{2\pi i}\int_{\Gamma_\infty}\frac{dg}{g} = \frac{1}{2\pi i}\int_{g(\Gamma_\infty)}\frac{dz}{z} = 2\,. \tag{18}$$

Hence from (18), $Q(k_1,\Phi_1)$ is the winding number of g. From the definition of charge and $u = -2i(\partial/\partial x)(\Phi_1 + \Phi_{\overline{1}})$, it follows that $Q(k_1,\Phi_1) = Q(\overline{k}_1,\Phi_{\overline{1}})$. One proceeds in a similar manner for higher charges. For the examples discussed earlier it is also clear that $Q = (\text{index}\ F)/2$ where F is defined by Eqs. (6b) or (9b), etc.

Full details will be published separately.

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