

Order Parameter and Scaling Fields in Self-Organized Criticality

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We present a unified dynamical mean-field theory for stochastic self-organized critical models. We use a single site approximation, and we include the details of different models by using effective parameters and constraints. We identify the order parameter and the relevant scaling fields in order to describe the critical behavior in terms of the usual concepts of nonequilibrium lattice models with steady states. We point out the inconsistencies of previous mean-field approaches, which lead to different predictions. Numerical simulations confirm the validity of our results beyond mean-field theory. [S0031-9007(97)03402-9]

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The origin of scaling in Nature [1] has become in recent years a challenging problem in physics. Bak, Tang, and Wiesenfeld (BTW) [2] have proposed self-organized criticality (SOC) as a unifying theoretical framework to describe a vast class of driven systems that evolve “spontaneously” to a stationary state, characterized by power law distributions of dissipation events. Despite the insights SOC concepts have brought to a number of problems, an agreement on the precise definition of SOC has still not been reached, and the exact meaning of the word “spontaneous” is quite unclear. Originally, SOC was associated with the absence of tuning parameters, but it has been noted [3] that the driving rate acts as a tuning parameter in most, if not all, SOC models. This ambiguity has hindered the formulation of precise relations between SOC and other nonequilibrium critical phenomena [4,5].

In this Letter we reformulate SOC in terms of typical concepts of nonequilibrium critical phenomena [5] by using the dynamical single site mean-field (MF) theory [6]. We provide a general scheme in which the details of different models are included via effective parameters and constraints. We mainly discuss sandpile models [2] with and without dissipation [7], but the formalism can be directly applied to other stochastic SOC models, such as the forest-fire model [8]. We find two independent critical parameters, i.e., relevant scaling fields, both with critical value equal to zero, and, just in this double limit, criticality is reached. We study the behavior of the order parameter and evaluate critical exponents. The results we obtain are in contrast with previous MF approaches [9,10]. This is due to a subtle inconsistency in the way critical parameters have been chosen in previous works. We show that some MF exponents are exact also in low-dimensional systems because of conservation laws. Our predictions are confirmed by numerical simulations of two-dimensional sandpile models.

Sandpile models are cellular automata with an integer (or continuous) variable z_i (energy) defined in a d -dimensional lattice. At each time step an energy grain is added to a randomly chosen site, until the energy of a site

reaches a threshold z_c . When this happens the site relaxes ($z_i \rightarrow z_i - z_c$), and energy is transferred to the nearest neighbors ($z_j \rightarrow z_j + y_j$). For conservative models the transferred energy equals the energy lost by the relaxing site ($\sum y_j = z_c$), at least on average. Usually, the only form of dissipation occurs at the boundary, from which energy can leave the system. With these conditions the system reaches a stationary state characterized by avalanches whose sizes s are distributed as a power law $P(s) \sim s^{-\tau}$ [2,11,12].

In order to simplify the description of these models, we can reduce the number of states each site can assume in the following way. We divide sites as *critical*, *stable*, and *active* [12]. Stable sites are those that do not become active if energy is added to them. Critical sites become active by the addition of energy. Active sites are relaxing and transfer energy, providing an interaction with other sites, usually the nearest neighbors (nn). In this way we have mapped the system in a three state cellular automaton (CA) on a d -dimensional lattice [5,13]. To each site, i is associated with a variable s_i , which can assume three different values. A complete set $s \equiv \{s_i\}$ of lattice variables specifies a configuration of the system. The dynamics is characterized by the operator $\langle s|W|s^0 \rangle$ which represents the transition rate from a configuration s_0 to a configuration s in a time step t . A well-established technique to study these systems is the single site mean-field approximation [6]. Denoting by ρ_a, ρ_c , and ρ_s the average densities of sites in the active, critical, and stable states, respectively, we write the following reaction rate equations:

$$\frac{\partial}{\partial t} \rho_\kappa = F_\kappa(\rho_a, \rho_c, \rho_s), \quad \kappa = a, c, s. \quad (1)$$

Because the densities must preserve normalization, two of the above equations supplemented with the condition $\rho_a + \rho_c + \rho_s = 1$ are enough to describe completely the system. The explicit form of Eq. (1) can be derived from the master equation of the model. This approach leaves room for a systematic treatment of higher order correlations, as is done in [6]. The derivation is straightforward but

lengthy, and it will be reported elsewhere [14]. In general, F_κ can be expanded as a series of the average densities:

$$F_\kappa = \sum_n f_\kappa^n \rho_n + \sum_{n,\ell} f_\kappa^{n,\ell} \rho_n \rho_\ell + \mathcal{O}(\rho_n^3), \quad (2)$$

where the constant term is set to zero in order to get a stationary state. The first order terms are the transition rates generated by the external driving fields or by spontaneous transitions. The second and higher order terms characterize transitions due to the interaction between different sites. In SOC models, only the active state generates a nontrivial dynamical evolution, while stable or critical sites can change their state only because of the external field or the presence of an active nn site. Since the critical point is identified by $\rho_a = 0$, in correspondence with a vanishing external field, we can neglect second order terms in the density of active sites. The solutions of the stationary equations ($\frac{\partial}{\partial t} \rho_\kappa = 0$) are functions of the effective parameters $f_\kappa^n, f_\kappa^{n,\ell}$, which depend on the details of the model. It is expected that the critical behavior is not affected by the specific values of the parameters, while universality classes will depend on constraints imposed on the equations, because of symmetries and conservation laws.

For the sake of clarity, we describe in detail the case of sandpile models. In this class of systems the only external field is the flow of energy added to the system. We can describe this driving by the probability per unit time h that a site will receive a grain of energy. The total amount of energy added to the system at each time step will be $J_{\text{in}} = hL^d$. The first order terms in F_a are the transition rates $a \rightarrow s, c$ and vice versa, independent of nearest neighbor sites,

$$f_a^a = -1, \quad f_a^s = 0, \quad f_a^c = h. \quad (3)$$

Here we considered that active sites become stable with a unitary rate, stable sites never become active, and critical sites become active because of the external field. In addition, there is a single interaction term that describes the creation of an active site from a critical site due to the relaxation of nn sites. We can write this term as $(g - \epsilon)\rho_c\rho_a$, where g is an effective rate that depends on the geometry and the energy involved in the relaxation process and ϵ is the average energy dissipated in each site [14]. We stress that ϵ is also present for fully conservative systems, being an effective term due to the boundary dissipation. Considering all these terms, we obtain

$$F_a = -\rho_a + h\rho_c + (g - \epsilon)\rho_c\rho_a + \mathcal{O}(\rho_a^2). \quad (4)$$

A similar reasoning yields the functions F_c and F_s . The effect of the driving field on stable sites and the interaction between active and stable sites deserve a discussion. The corresponding terms contribute to the transition rate $s \rightarrow c$. In sandpile models, the energy conservation imposes a local constraint in the rate equations. Energy is stored in stable sites until they become critical, but only a fraction u of stable sites receiving an energy grain contributes to the $s \rightarrow c$ process. Therefore, in this case, the reaction rates will be given by the $c \rightarrow a$ rates multiplied by the factor u .

For instance, a stable site will receive an energy grain with probability h , but it will turn critical only with probability uh . The same reasoning holds for the interaction term.

After imposing stationarity, we get the following dynamical MF equations:

$$\begin{aligned} \rho_a &= h\rho_c + (g - \epsilon)\rho_c\rho_a, \\ \rho_a &= uh\rho_s + u(g - \epsilon)\rho_s\rho_a, \\ \rho_a &= 1 - \rho_s - \rho_c, \end{aligned} \quad (5)$$

where u, g are effective parameters which depend upon the particular model, ϵ represents the dissipation, and h is the driving field. We expect g to be an independent parameter of the model, while u has to be obtained self-consistently because it is fixed once the dynamical rules of the CA are given.

After some algebra from Eqs. (5), we obtain a closed equation for ρ_a ,

$$u(g - \epsilon)\rho_a^2 + [1 + u(1 + h - g + \epsilon)]\rho_a - uh = 0. \quad (6)$$

We can expand $\rho_a(h)$ for small values of the field h . The zero order term in the expansion vanishes and we obtain a leading linear term,

$$\rho_a(h) = \frac{uh}{1 + u - ug + u\epsilon}. \quad (7)$$

This result has to be consistent with the global conservation law, which states that the average input energy flux J_{in} must balance the dissipated flux J_{out} . In the stationary state the conservation law can be written as

$$J_{\text{in}} = hL^d = J_{\text{out}} = \epsilon\rho_a L^d. \quad (8)$$

By comparing Eq. (7) with Eq. (8) we obtain that $u = 1/(g - 1)$. In the limit $h \rightarrow 0$ the densities are therefore given by

$$\begin{aligned} \rho_a &= \frac{h}{\epsilon}, \quad \rho_c = \frac{1}{g} + \mathcal{O}(h), \\ \rho_s &= \frac{g - 1}{g} + \mathcal{O}(h). \end{aligned} \quad (9)$$

An estimate of g can be obtained using a random neighbor approximation, which yields $g = 2d$ for the BTW model [2] or $g = 2$ for the two level models [12]. Noticeably, in the latter case, $u = 1$, i.e., all stable sites are subcritical, as is expected for a two level model.

We now discuss the critical behavior of these systems. The balance between conservation laws and dissipation is essential for the critical behavior of the model, as also pointed out in [15]. The model is critical just in the double limit $h, \epsilon \rightarrow 0, h/\epsilon \rightarrow 0$, similar to the forest-fire model [8]. In analogy with nonequilibrium phenomena [5,13], the one particle density of active sites is the *order parameter* and goes to zero at the critical point. We can then distinguish several different regimes as a function of the parameters. The system has no stationary state for $h > \epsilon$, since ρ_a would have to be greater than one to

satisfy Eq. (8). The model is supercritical for $h > 0$ and $\epsilon > h$, while for $h \rightarrow 0$ and $\epsilon > 0$ it is subcritical and the dynamics displays avalanches. The phase diagram is somehow similar to that of usual second order phase transitions, if we replace h by the magnetic field and ϵ by the reduced temperature.

In the supercritical regime the order parameter is linear in h ,

$$\rho_a \sim h^{1/\delta}, \quad \delta = 1. \quad (10)$$

This is analogous to the MF results obtained for contact processes and other nonequilibrium CA [5,6,13], but it is in contrast with previous MF approaches for sandpile models [9,10], which yielded $\delta = 2$. This incorrect result is due to an inconsistency present in those studies. The scaling is expressed in terms of the average energy $\theta \equiv \sum_i \rho_i z_i$ which is treated as an independent control parameter. As we have just shown, θ and h are not independent. Moreover, θ cannot be considered as the control parameter even for $h = 0$, since it does not determine completely the state of the system: The same value of θ describes several states corresponding to different values of densities ρ_i . This is a typical property of CA with multiple absorbing states [13]. In analogy with nonequilibrium CA it is possible to define several other exponents characterizing the supercritical regime [14].

In the subcritical regime, the behavior of the system is dominated by the dissipation. This can be seen by studying the susceptibility,

$$\chi \equiv \frac{\partial \rho_a(h)}{\partial h} = \frac{1}{\epsilon}, \quad (11)$$

which diverges for $\epsilon = 0$. The system is in a subcritical state for any value of ϵ different from zero. The critical behavior is thus characterized by the scaling laws $\chi \sim \epsilon^{-\gamma}$ and $\xi \sim \epsilon^{-\nu}$, where ξ is the characteristic length.

We can use these exponents to characterize the conservative sandpile model, since our MF analysis treats both boundary and bulk dissipation. In conservative systems, when the size is increased, the effective dissipation depends on the system size. We can therefore assume that $\epsilon \sim L^{-\mu}$. At the same time, the characteristic length of the avalanches should go the same as $\xi \sim L$ to ensure dissipation of energy. This implies that the scaling relation $\nu\mu = 1$ and that $\chi \sim L^{\mu\gamma}$. It is also possible to show [14] that the susceptibility scales as the average avalanche size, and in two dimensions it has been found that $\langle s \rangle \sim L^2$ exactly for $L \rightarrow \infty$ [16]. The same result also holds in MF theory if we assume that the dynamics is diffusionlike [14,17,18].

Combining all the above results, we obtain a first set of MF exponents,

$$\gamma = 1, \quad \mu = 2, \quad \nu = 1/2. \quad (12)$$

It is worthwhile to remark that it is not possible to define the equivalent of an exponent β because for $h = 0$ the order parameter is always zero. We emphasize again that h and ϵ are both control parameters responsible for different regimes of the model.

We have derived these exponents using only conservation laws, therefore we expect they should also describe low-dimensional sandpile models. We simulate numerically the BTW model with finite driving rate h and boundary dissipation. We see in Fig. 1 that the density of active sites goes to zero linearly with h ($\delta = 1$) with a slope that increases with the system size as L^2 . This is in agreement with the MF theory which predicts that the susceptibility scales as $L^{\mu\gamma}$, with $\mu\gamma = 2$. To observe more clearly the scaling with dissipation of the sandpile model, we study the BTW model with *periodic* boundary conditions and fixed dissipation ϵ [19]. In Fig. 2 we plot the control parameter as a function of h/ϵ . The scaling predicted by the MF theory ($\gamma = 1$) is verified with remarkable accuracy, and we note that finite size corrections are not noticeable, in contrast with the case of boundary dissipation [11]. Finally, the exponent $\nu = 1/2$ has already been measured in a two-dimensional dissipative sandpile model [7].

The dynamics in the subcritical regime takes place in the form of avalanches. The exponents describing avalanche distributions, in general, will not agree with the MF results and have to be calculated by the renormalization group [20]. A complete characterization of MF avalanche scaling has been obtained by using the theory of branching processes [21]. Here, we reproduce these results in an independent way. Following Grassberger and de la Torre [5], we consider the probability that a small perturbation activates s sites (an avalanche in the SOC terminology),

$$P(s, \epsilon) = s^{-\tau} \mathcal{G}(s/s_c(\epsilon)), \quad (13)$$

where $s_c \sim \epsilon^{-1/\sigma}$ is the cutoff in the avalanche size. The perturbation decays in the stationary subcritical state as

$$\rho_a(t) \sim t^\eta \mathcal{F}(t/t_c). \quad (14)$$

Here t_c denotes the characteristic time which scales as $t_c \sim \epsilon^{-\Delta}$. We can obtain these exponents by solving

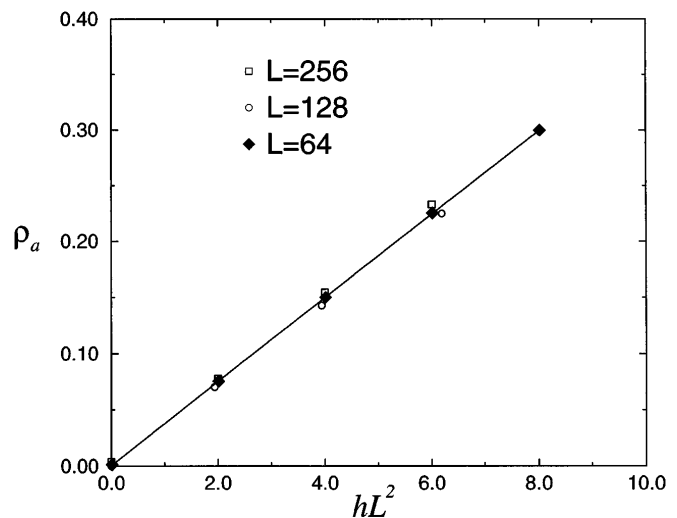


FIG. 1. The density of active sites in the BTW model with boundary dissipation as a function of the driving rate h .

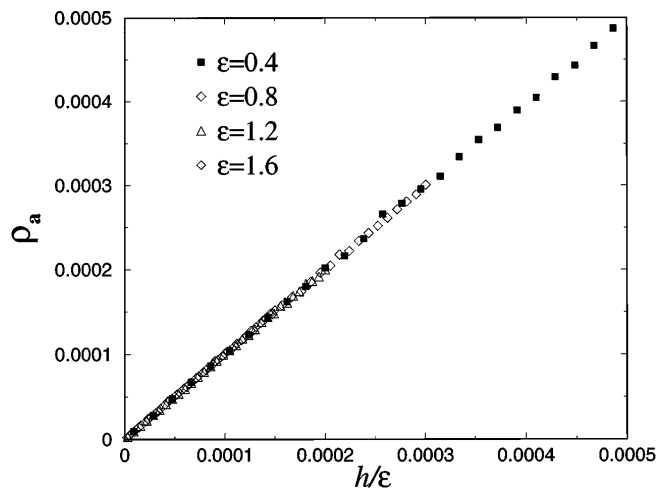


FIG. 2. The density of active sites for the BTW model with bulk dissipation ϵ and periodic boundary conditions as a function of h/ϵ for $L = 64$.

Eq. (1) for a small perturbation around the subcritical stationary state ($h = 0$),

$$\frac{\partial \rho_a}{\partial t} = -\rho_a + (g - \epsilon)\rho_a\rho_c, \quad (15)$$

where, as a first order approximation, we can replace $\rho_c = 1/g$. We obtain in this way $\rho_a(t) \sim \exp(-\epsilon t/g)$ which implies $\eta = 0$ and $\Delta = 1$. Introducing the scaling laws $s_c \sim \xi^D$ and $t_c \sim \xi^z$, it is possible to derive [14] another set of scaling relations,

$$\gamma = \frac{(2 - \tau)}{\sigma}, \quad D = \frac{1}{\nu\sigma}, \quad z\nu = \Delta, \quad (16)$$

$$\frac{(\tau - 1)}{\nu\sigma} = z,$$

from which we get the second set of MF critical exponents,

$$z = 2, \quad D = 4, \quad \tau = 3/2, \quad \sigma = 1/2, \quad (17)$$

in agreement with the theory of branching processes [21]. It is worthwhile to remark that the numerical value of these exponents is the same as in other MF approaches [9,10], but their significance is completely different, being defined with respect to a different scaling field.

We have obtained a complete characterization of the critical properties of the sandpile model. The critical state arises due to the fine-tuning of the driving rate and the dissipation. This condition is enforced implicitly in the BTW model by imposing time scale separation and dissipation only through the boundaries, which makes h and ϵ equal to zero in the thermodynamic limit. In this formalism, SOC appears as a special case of nonequilibrium critical phenomena, with the only peculiarity being that the critical parameters are zero. The same MF analysis applied to the forest-fire model leads to similar conclusions [14,22], but the absence of conservation laws implies that MF exponents are not correct in low dimensions. We hope this

will clarify the precise significance of SOC in the framework of nonequilibrium critical phenomena.

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