Detecting Unstable Periodic Orbits of Chaotic Dynamical Systems

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A method to detect the unstable periodic orbits of a chaotic dynamical system is developed. For a given dynamical system our approach allows us to locate the unstable periodic cycles of, in principle, arbitrary length with a high accuracy. Preknowledge of the dynamical system is not required. To demonstrate its reliability as well as efficiency we apply it to several two-dimensional chaotic maps. In the case of short chaotic time series we develop a dynamical algorithm which is based on a mean-field approach via the Voronoi diagram belonging to the time series. This algorithm enables us to detect low period cycles using a very small set of data points. The influence of noise is investigated in some detail. [S0031-9007(97)03430-3]

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It is a well established fact that the chaotic behavior of low dimensional dynamical systems is to a large extent determined by the position and the stability properties of the unstable periodic orbits existing in the chaotic sea [1]. In particular, fractal dimensions, Lyapunov exponents, invariant measures, etc. of chaotic attractors can be expressed in terms of the periodic cycles [1,2]. Moreover the quantum mechanical properties of classically chaotic conservative systems possess, in the semiclassical regime, a series expansion with respect to the lengths and the stability coefficients of the periodic orbits. Since chaotic behavior is inherent to many different dynamical systems periodic orbit theory possesses numerous applications and is of relevance to different areas of physics.

Finding the unstable periodic orbits in the chaotic sea is, however, in practice, even for rather simple dynamical systems, a difficult task. It usually requires a high numerical effort and bears also methodical problems. This is the case if the dynamical law is given, but it is even more striking in situations where the only available information about the system is an experimentally observed time series. One has then often to extract all the properties of the unstable periodic orbits of the underlying dynamical law by using a small set of data. It is therefore not surprising that many recent works deal with the development of efficient methods for the detection of the periodic orbits both for a given dynamical law [3–5] as well as an experimental time series [6–8].

In the present Letter we provide a new approach to this problem. The basic ingredient of our method is a suitable transformation applied to the discrete dynamical law (or to the corresponding time series) in order to obtain a new system in which the unstable fixed points of the original system are stable but retain their original positions in space. It turns out that this transformation possesses an appealing geometrical interpretation in terms of a vector field which is organized through the positions of the periodic cycles (see below). The progress within our treatment is twofold: first we present a very efficient method to calculate periodic orbits of, in principle, *arbitrary length* for a given discrete N-dimensional dynamical system with *any desired accuracy*. No restriction in the dimension of the system is assumed. Second, given a multidimensional fully chaotic very short time series we develop a mean-field approach which allows us to determine the positions of the lowest unstable cycles. The corresponding algorithm is stable with respect to noise and works remarkably well even in cases where the number of points in the time series is extremely small (~100 points for the Ikeda map), i.e., for cases where the linear neighborhood of the cycles is not visited at all.

Let us consider a N-dimensional discrete fully chaotic dynamical system given by

$$U: \quad \vec{r}_{i+1} = \vec{f}(\vec{r}_i)\,. \tag{1}$$

U, being fully chaotic, possesses only unstable fixed points. Our goal is to construct from Eq. (1) other dynamical systems S_k with the same number of fixed points which are still at their original positions but have become stable through the transformation L_k: $U \to S_k$. The transformation L_k changes therefore the stability properties but not the location of the fixed points. If we succeed with our plan then the search for the fixed points (periodic cycles) of the system U becomes a trivial task: because of the stability of the fixed points in the constructed system S_k every trajectory of S_k runs after some iterations to a fixed point $\vec{r}_F$. Per construction $\vec{r}_F$ is then also a fixed point of the system U. To fulfill the requirement of the one to one correspondence between the fixed points of U and S_k the transformation L_k should in general be linear. Consequently S_k takes on the following appearance:

$$S_k: \quad \vec{r}_{i+1} = \vec{r}_i + \mathbf{\Lambda}_k(\vec{f}(\vec{r}_i) - \vec{r}_i)\,, \tag{2}$$

where $\mathbf{\Lambda}_k$ is an invertible $N \times N$ constant matrix. It is straightforward to show that the above definition (2) satisfies the one to one correspondence of the fixed points of U and those of S_k. The dynamical laws U and S_k possess fixed points at identical positions in space.

Our next step is to stabilize the fixed points of the transformed systems S_k by suitable choices of $\boldsymbol{\Lambda}_k$. In general different fixed points are then stable in different transformed systems S_k according to the different matrices $\boldsymbol{\Lambda}_k$. It turns out that if the absolute values of the elements of the matrices $\boldsymbol{\Lambda}_k$ are sufficiently small ($|\lambda_{ij}| \ll 1$, $i, j = 1, \ldots, N$) then there exists a universal set of very restrictive matrices such that at least one matrix belonging to this set transforms [via Eq. (2)] a given unstable fixed point of U to a stable fixed point of the corresponding S_k. In order to determine this set of matrices let us consider the stability matrices $\boldsymbol{T}_U$ and $\boldsymbol{T}_{S_k}$ of U and S_k which obey the following relation:

$$M_k: \quad \boldsymbol{T}_{S_k} = \boldsymbol{1} + \boldsymbol{\Lambda}_k(\boldsymbol{T}_U - \boldsymbol{1}). \tag{3}$$

For $\boldsymbol{T}_U$ we assume that it is real, invertible, and diagonalizable. Since $\vec{r}_F$ is an unstable fixed point at least one of the eigenvalues of $\boldsymbol{T}_U$ at $\vec{r}_F$ must possess an absolute value greater than 1. In order to stabilize $\vec{r}_F$ we proceed in two steps: first we use the parametrization $(\boldsymbol{\Lambda}_k)_{ij} = (\lambda \boldsymbol{C}_k)_{ij}$ with $1 \gg \lambda > 0$ and $C_{ij} = O(1)$. The matrix C_{ij} has to be chosen such that the real parts of all eigenvalues of the matrix $\boldsymbol{C}_k \cdot (\boldsymbol{T}_U - \boldsymbol{1})$ are negative. If this is achieved then the next step is to use a sufficiently small value for the parameter λ, such that the eigenvalues of the matrix $\boldsymbol{1} + \lambda \boldsymbol{C}_k(\boldsymbol{T}_U - \boldsymbol{1})$ have absolute values less than 1. It is obvious that this can always be achieved if λ is sufficiently small.

It can be shown that it is always possible to find a involutory matrix $\boldsymbol{C}_k$ ($\boldsymbol{C}_k^2 = \boldsymbol{1}$) such that $\boldsymbol{C}_k(\boldsymbol{T}_U - \boldsymbol{1})$ has eigenvalues with negative real parts. In fact, in practice it turns out that an even more restricted form for $\boldsymbol{C}_k$ is sufficient to achieve stabilization, namely that all the matrices correspond to special reflections in space. The elements of the matrices $\boldsymbol{C}_k$ are then $C_{ij} \in \{0, \pm 1\}$ and each row or column contains only one element which is different from zero. The matrices $\boldsymbol{C}_k$ are therefore orthogonal. The number a_N of such matrices in N-dimensional space is given by $a_N = N!2^N$ (for a more elaborate discussion, see [9]). Using an appropriate $\boldsymbol{C}_k$ out of the above-defined set and a sufficiently small value for λ we always succeed in making stable a chosen fixed point of U. An important fact is that each matrix of the set $\{\boldsymbol{C}_k\}$ stabilizes not only a single fixed point (periodic orbit) but a whole rather general class of periodic orbits ranging up to arbitrarily high periods. In two dimensions, for example, any hyperbolic fixed point with reflection belonging to arbitrarily high iterates can be stabilized by using for $\boldsymbol{C}_k$ the unit matrix. A more detailed account of this subject goes beyond the scope of the present paper and will be given elsewhere [9]. The most important advantage of the stabilization process is its global character. Even points lying far from the linear neighborhood of the stabilized fixed point are attracted to it after a few iterations of the transformed dynamical law. Because of the fact that different kinds of fixed points (e.g., hyperbolic with or without reflection) are stabilized by different matrices $\boldsymbol{C}_k$ our stabilization procedure offers the possibility to distinguish in the stabilization process among the different types of fixed points.

Having presented the basic features of our method we will discuss now briefly how it can be used to detect the unstable periodic orbits of a given N-dimensional discrete fully chaotic dynamical system. Using Eq. (2) we transform the given system U into a new system. For the matrix $\boldsymbol{\Lambda}_k = \lambda \boldsymbol{C}_k$ we use a sufficiently small value of λ and $\boldsymbol{C}_1$ from $\{\boldsymbol{C}_k\}$, $k = 1, \ldots, a_N$. Iterating an arbitrary starting point $\vec{r}_0$ with S_1 from Eq. (2) forward in time the corresponding trajectory either arrives after a few steps at a fixed point $\vec{r}_F$ or the corresponding trajectory escapes to infinity. Subsequently we perform the same procedure for the next matrix $\boldsymbol{C}_2$ of the set $\boldsymbol{C}_k$, etc. until all the a_N matrices $\boldsymbol{C}_k$ have been used. In order to find the unstable points of the periodic cycles of period i we simply have to replace $\vec{f}$ in Eq. (2) by its ith iterate $\vec{f}^{(i)}$. Following the above procedure it turns out that the number of starting points needed to obtain all periodic orbits of a given period on, for example, an attractor (see below) is only a few times more than the number of periodic orbits themselves. Suitable starting points can be obtained by taking, for example, a chaotic trajectory of the original system U. There exists a simple but nevertheless reliable strategy in order to ensure that all the periodic orbits of a given period have been detected. Assuming that a certain number of periodic orbits of period p have been found we just have to iterate a multiple of the starting points used in the previous run. If no additional periodic orbits show up the number of detected orbits can safely be assumed to be complete.

We applied our method to several 2D iterative maps like, for example, the Henon map, 2D logistic map, and, in particular, the more complicated Ikeda map which is given by $x_{n+1} = 1 + 0.9(x_n \cos w_n - y_n \sin w_n)$, $y_{n+1} = 0.9(x_n \sin w_n + y_n \cos w_n)$, and $w_n = 0.4 - \frac{6}{1+x_n^2+y_n^2}$. The method turns out to be fast and extremely accurate. To demonstrate the reliability and efficiency of our method we have calculated all unstable periodic orbits of the Ikeda attractor, i.e., their location and stability eigenvalues, up to period 13 with a relative accuracy of 10^{-14}. The number of cycles with period 3, 5, 7, 9, 11, and 13, for example, is 2, 4, 10, 26, 76, and 194, respectively. As an application we have calculated the topological entropy of the attractor and obtained 0.602. We emphasize that period 13 is by no means the maximal period which can be investigated but was chosen for reasons of demonstration.

The transformed dynamical law (2) possesses an appealing geometrical interpretation. Let $\{\vec{r}_j,\ j = 1, \ldots, m\}$ be a trajectory of the system U. At each point of the trajectory we define a vector field $\vec{V}_U(\vec{r}_j) = \vec{r}_{j+1} - \vec{r}_j$. The corresponding transformation L_k represents then a reflection of each vector $\vec{V}_U(\vec{r}_j)$ combined with a subsequent scale transformation of its length with the factor λ. The resulting transformed vector field $\vec{V}_{S_k}$ is then organized globally around those fixed points which have been stabilized. The flow of the vector field $\vec{V}_{S_k}$ is, therefore, centered and organized by the positions of the stable fixed points. The fact that the stabilized fixed points represent the centers of the flow of the vector field $\vec{V}_{S_k}$ possesses a counterpart in the original chaotic system U: the chaotic trajectory exhibits turning points in any direction of phase space in the vicinity of the fixed point [10].

To illustrate the above-discussed properties we show in Fig. 1 the vector fields belonging to a chaotic trajectory (100 points) on the attractor of the Ikeda map. In Fig. 1(a) the vector field $\vec{V}_U(\vec{r}_j)$ is illustrated. The above-mentioned turning point property around the fixed point $\vec{r}_F = (0.53, 0.25)$ can clearly be seen. In Fig. 1(b) the corresponding vector field $\vec{V}_{S_k}$ is illustrated. (For the fixed point of the Ikeda map the stabilizing matrix is $\boldsymbol{C}_k = \mathbf{1}$.) The global organization of the flow towards the fixed point $\vec{r}_F$ is evident. Obviously this property is not restricted or specific for the linear neighborhood of the fixed point but provides a global feature of the dynamical system.

These properties provide an excellent starting point for an application of our stabilization method to the analysis of time series. The important feature allowing us to apply our stabilization method to the analysis of time series is the following: to every stability matrix $\boldsymbol{T}_U$ belonging to an unstable fixed point of the original dynamical system there exists a matrix $\boldsymbol{C}_n$ belonging to the above-defined universal set which, applied to the dynamical law or alternatively to the data of the time series, transforms this fixed point to a stable one. To achieve stabilization of a fixed point *we therefore do not assume the knowledge of* $\boldsymbol{T}_U$ *at this fixed point:* at least one of the matrices of the set $\{\boldsymbol{C}_k\}$ will cause stabilization of the unstable fixed point with stability matrix $\boldsymbol{T}_U$.

The global character of the organization of the flow will be helpful in extracting the approximate positions of the fixed points within a very small set of data. In the case of very short time series the linear neighborhood of the fixed points is frequently not visited at all by the finite trajectory. Methods relying on the properties of the dynamics in the linear neighborhood of the fixed point are therefore expected to fail with respect to the determination of the fixed points (periodic orbits).

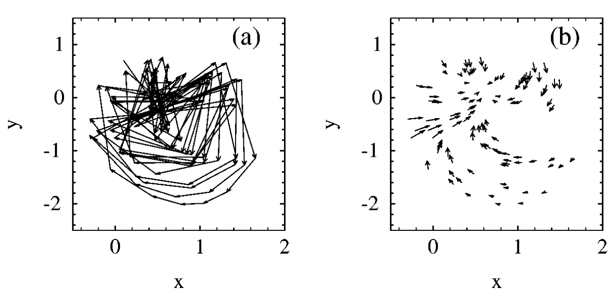

FIG. 1. (a) The vector field $\vec{V}_U$ trajectory consisting of 100 points on the Ikeda attractor. (b) The corresponding vector field $\vec{V}_{S_k}$. (See text.)

We begin our analysis by considering a finite multidimensional time series $\{\vec{r}_i,\ i = 1, \ldots, m\}$. We do not address here the question of the extraction of a multidimensional time series from a one-dimensional experimental signal but instead refer the reader to the standard reconstruction methods existing in the literature [11]. We emphasize that our goal is to find the unstable fixed points in a very short chaotic (and noisy) time series [8] and not to extract as many periodic orbits as possible from a long time series. To achieve this we have developed a dynamical algorithm consisting of three main steps.

First we construct the so-called Voronoi diagram belonging to the set of points of a given time series. The Voronoi diagram is the union of all the Voronoi zones. For each point of the time series we construct its corresponding Voronoi zone. The Voronoi zone belonging to the point $\vec{r}_j$ is defined as the set of points which possess a smaller distance with respect to $\vec{r}_j$ than to any other point of the time series. Subsequently the above-described stabilization procedure is used to create the vector field $\vec{V}_{S_k}$ associated with the time series. As a next step we apply a mean-field approximation in order to obtain a vector field on the complete Voronoi diagram: to each point belonging to a certain Voronoi zone we assign the same vector belonging to the point of the time series of this zone. In this way we construct a vector field defined in the entire embedding space.

We then randomly choose a point belonging to the Voronoi diagram and start iterating this point in the following manner: each iteration is a translation by the vector belonging to the corresponding point. After a number of steps the resulting trajectory has moved forward to the neighborhood of the stabilized fixed point. The trajectory is then trapped by a few Voronoi zones. We call the resulting set of data points corresponding to these Voronoi zones a dynamically invariant set.

The determination of the coordinates of the fixed point takes place in the third step. Knowing the dynamically invariant set we proceed with the iterations using an adiabatic scaling procedure: every $m(\gg 1)$ iterations the vector field is rescaled to smaller and smaller values (typically $m \approx 100$). Finally the trajectory converges to the common points of the Voronoi zones (the intersection point of their boarder lines) of the invariant set. Its coordinates provide an excellent approximation for the position of the fixed point.

In the following we apply this algorithm to the case of the Ikeda map given above. Our time series consists of a trajectory of *only 100 points* on the corresponding attractor. We emphasize that this trajectory does not visit the linear neighborhood of the fixed point at all and therefore any method using properties of the linear neighborhood in order to detect the unstable fixed points cannot be successful. Using the above algorithm the result for the fixed point is $\vec{r}_F = (0.536, 0.225)$ which should be compared with the exact value $\vec{r}_F = (0.53275, 0.24690)$. A graphical illustration of a typical path resulting from our algorithm is illustrated in Fig. 2. Using 200 points of the same trajectory we obtain for the period two cycle $(0.52184, -0.55542)$; $(0.57946, 0.51272)$ compared to the exact values $(0.50984, -0.60837)$; $(0.62160, 0.60593)$.

We studied the influence of noise on the stability of our results. Using the noisy data set $\vec{x}_i^{\text{noisy}} = \vec{x}_i + \epsilon\vec{\xi}_i$, where $\vec{x}_i$ represents the data set without noise, we varied the parameter ϵ from 0.1 to 0.3 ratios of root means square amplitudes of the attractor. We chose two different kinds of noise for the variable $\vec{\xi}_i$: (a) uniform noise in $(-1, +1)$ and (b) Gaussian distributed noise with zero mean and a variance of 0.2. To improve the statistics and to avoid the effects of special configurations we applied our algorithm to 100 randomly chosen starting points on the attractor. For the case of $\epsilon = 0.1$ and 0.3 rms we found a relative accuracy of the position of the fixed point of 4.5% and 16.1%, respectively, for uniform noise and 4.0% and 6.3%, respectively, for Gaussian noise.

We have presented a new method to detect unstable periodic orbits in a fully chaotic discrete dynamical system. The most important advantage of our approach is its global character. If the dynamical law is given our method allows us to extract, in principle, arbitrarily high periodic orbits with a very high accuracy. We mention that our approach is not restricted to discrete dynamical systems. Periodic orbits in continuous dynamical systems can be detected by using the Poincaré map which is a discrete map representing the original continuous dynamical system in a chosen hyperspace. Furthermore, based on this stabilization method, we developed a dynamical algorithm which allows us to detect unstable periodic orbits in a multidimensional chaotic time series using a very small set of (experimental) data points. This may be helpful for the analysis of biological systems where in many cases the number of measurements is, due to experimental reasons, extremely limited.

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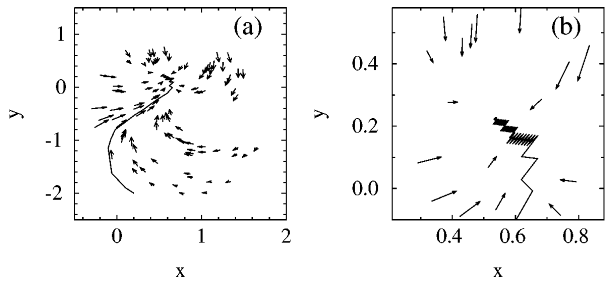

FIG. 2. (a) The path resulting from the application of the dynamic algorithm to a time series consisting of 100 data points of the Ikeda attractor. (b) A zoom of (a) in the neighborhood of the fixed point. The invariant set as well as the adiabatic scaling are clearly seen.

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