Phase and Phase Diffusion of a Split Bose-Einstein Condensate

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We analyze theoretically an experiment in which a trapped Bose-Einstein condensate is cut in half, and the parts are subsequently allowed to interfere. If the delay between cutting and atom detection is small, the interference pattern of the two halves of the condensate is the same in every experiment. However, for longer delays the spatial phase of the interference shows random fluctuations from one experiment to the other. This phase diffusion is characterized quantitatively. [S0031-9007(97)03441-8]

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Progress in experiments with weakly interacting Bose-Einstein condensates in alkali vapors [1–4] has spurred a vigorous interest in the phase of the condensate. In condensed-matter physics it is routinely assumed that, in the process of spontaneous breaking of gauge symmetry, a condensate picks up a phase akin to the phase of the electric field of laser light [5,6]. The phase should lead to spatial interference of two condensates [7–10], and analogs of the Josephson effect [11,12]. Recently it has transpired that the assumption of spontaneous symmetry breaking is unnecessary for the explanation of these effects. Instead, a measurement looking for interference of two condensates will find the characteristic consequences of the phase, even if there is no phase in the initial state of the system [7,9,13–16]. Just as the phase of laser light diffuses, it is also becoming clear that the observed phase of the condensate is subject to random time evolution [17–21]. Here the analog to lasers is not entirely accurate, though, because in a condensate the mechanism of phase diffusion is the interactions between the atoms.

Experimental observations of spatial interference of two condensates have been reported by the MIT group [3]. Their key technical innovation is the capability to split the magnetic trap holding a condensate into two by making use of the dipole forces of far-off resonant laser light. The question naturally arises about the difference of the outcome of an experiment depending on whether the trap is first split and evaporative cooling is subsequently applied to produce two condensates, or if the condensate is formed first and then cut. The MIT experiments [3] were of the former variety, while we in this Letter address the latter "cool-cut-interfere" scheme. We show that the two halves of the split condensate start out with a fixed relative phase between them: the interference pattern is the same every time the experiment is repeated. On the other hand, a setup for quantitative studies of phase diffusion arises when the condensate halves are allowed to evolve between the splitting and the measurement of interference. We present quantitative predictions for such an experiment. In particular, after an initial period, the standard deviation of the measured phase grows linearly in time at a rate we are about to determine. Our entire analysis is carried out without assuming spontaneous breaking of gauge symmetry.

We first consider a single atom in a one-dimensional infinite potential well in the interval $x \in (-L/2, L/2)$. The normalized wave functions $\psi_n(x)$ with $n = 0, 1, \ldots$, which we order according to ascending energy, are well known. Suppose that next a delta function potential barrier $V(x) = \alpha\,\delta(x)$ is erected at $x = 0$. An elementary analysis shows that in the limit $\alpha \to \infty$ the odd wave functions $\psi_1, \psi_3, \ldots$ remain untouched, while the even wave functions $\psi_0, \psi_2, \ldots$ develop an additional node at the origin. In fact, the new wave functions χ_n may be chosen as $\chi_{2n+1} = \psi_{2n+1}$ and $\chi_{2n}(x) = \mathrm{sgn}(x)\psi_{2n+1}(x)$, where sgn is the signum function. The states also become doubly degenerate: $\varepsilon_{2n+1} = \varepsilon_{2n}$. It is therefore possible to choose another set of normalized wave functions $\chi_n^{\pm} = \frac{1}{\sqrt{2}}(\chi_{2n} \pm \chi_{2n+1})$, degenerate for each n, in such a way that the wave function χ_n^+ (χ_n^-) is only nonzero for $x > 0$ ($x < 0$). This is from a microscopic point of view how the splitting of the trap works. Degenerate states are created that correspond to the atom being either entirely to the "left" ($x < 0$) or to the "right" ($x > 0$) of the barrier.

Continuing with the preceding example, suppose that there initially are N noninteracting bosons in the ground state of the potential well. The corresponding many-body wave function is $\psi(x_1, \ldots, x_N) = \psi_0(x_1)\psi_0(x_2)\ldots\psi_0(x_N)$. If the barrier is erected adiabatically, over a time scale long compared to the inverses of the excitation frequencies of the potential well, each one-particle wave function ψ_0 obviously evolves into χ_0. By virtue of the Bose symmetry, the many-body wave function turns into $\chi(x_1, \ldots, x_N) = \chi_0(x_1)\chi_0(x_2)\cdots\chi_0(x_N)$. Besides, the one-body wave function χ_0 may be represented as a superposition of two states $\chi_0^{\pm}$, each of which is confined to half of the trap: $\chi_0 = \frac{1}{\sqrt{2}}(\chi_0^+ + \chi_0^-)$. The many-body wave function may correspondingly be written

$$\chi(x_1,\ldots,x_N) = \frac{1}{2^{N/2}}[\chi_0^+(x_1) + \chi_0^-(x_1)] \cdots [\chi_0^+(x_N) + \chi_0^-(x_N)] = \frac{1}{2^{N/2}}\sum_{k=0}^{N}\sqrt{\binom{N}{k}}\,\chi_{k,N-k}^{-+}(x_1,\ldots,x_N), \quad (1)$$

where $\chi_{k,N-k}^{-+}$ stands for a unit-normalized and properly symmetrized boson wave function with k atoms in the state χ_0^- and $N-k$ atoms in the state χ_0^+. When the left and right states $\chi_n^{\pm}$ are chosen as the basis for second quantization, the state vector is

$$|\chi\rangle = \frac{1}{2^{N/2}}\sum_{k=0}^{N}\sqrt{\binom{N}{k}}|k, N-k\rangle, \quad (2)$$

where $|k, N-k\rangle$ stands for a number state with k atoms in the ground state χ_0^- of the left half of the potential well and $N-k$ atoms in the ground state χ_0^+ of the right half.

The infinite delta function barrier isolates the halves of the potential well from one another, and may thus be thought of as cutting the well into two separate physical systems. From this angle, the splitting has generated an entangled state of the two halves.

Next suppose that both the potential well and the delta function barrier are removed instantaneously, whereupon the atoms begin ballistic evolution, and at some later time t the positions of the atoms are detected. As far as the time evolution of a free atom is concerned, the one-body wave function $\chi_0(x_1)$ evolves into $\chi_0(x_1;t)$, and similarly for $x_2,\ldots,x_N$. Because the Bose symmetry is preserved, the total wave function at t again is simply $\chi(x_1,\ldots,x_N;t) = \chi_0(x_1;t)\cdots\chi_0(x_N;t)$. On the other hand, we have earlier proposed a detection theory for atoms along the lines of standard photon detection theory [7]. It may be seen easily that within this approach the joint probability density for detecting an atom at $x_1,\ldots,x_N$ is equal to the absolute square of the many-body wave function. In the present case the probability density is

$$P(x_1,\ldots,x_N;t) = |\chi_0(x_1;t)|^2\cdots|\chi_0(x_N;t)|^2. \quad (3)$$

Equation (3) implies that the atoms are detected independently of one another, in whatever interference pattern the wave function $\chi_0 = \frac{1}{\sqrt{2}}(\chi_0^+ + \chi_0^-)$ has evolved into. Our thorough experiment constitutes an N-fold repetition of Young's double slit experiment all at once, $\chi_0^{\pm}$ being the two waves that interfere. The spatial phase of the interference is the same in every run of the experiment. This should be contrasted with the case of two number-state condensates, for instance, if the initial state of the expansion were $|\chi\rangle = |N/2, N/2\rangle$ instead of (2). Then there would still be an interference pattern, but with a spatial phase that varies at random from one run of the experiment to the other [7,13,14].

The wave functions $\chi_0^{\pm}$ are both half-period pieces of a sine wave, so the double-slit interference pattern is rather trivial. How the interference arises in the actual experiments [3] is discussed in Refs. [8–10]. Here we digress with a toy model. We assume that at the same time as the atoms are released, the halves of the trap are launched toward one another; the atoms on the left are given a momentum translation to the right with the wave number per particle $\kappa \gg 1/L$, and each atom on the right receives the momentum translation $-\hbar\kappa$ The relevant left and right one-atom wave functions are $\tilde\chi_0^- = \chi_0^- e^{i\kappa x}$ and $\tilde\chi_0^+ = \chi_0^+ e^{-i\kappa x}$. When the ballistic atom clouds overlap, they evidently develop an interference pattern of the form $\cos(2\kappa x)$ superimposed on a slowly varying background. Suppose now that the left atoms are somehow given an additional phase shift φ, so that the initial one-atom wave function becomes $\tilde\chi(\varphi) = \frac{1}{\sqrt{2}}(\tilde\chi_0^- e^{i\varphi} + \tilde\chi_0^+)$. The interference pattern then shifts, and is proportional to $\cos(2\kappa x + \varphi)$. On the other hand, the state vector analogous to (2) becomes

$$|\tilde\chi(\varphi)\rangle = \frac{1}{2^{N/2}}\sum_{k=0}^{N} e^{i\varphi k}\sqrt{\binom{N}{k}}|k, N-k\rangle, \quad (4)$$

where second quantization now is with respect to the states $\tilde\chi_0^{\pm}$. This example graphically demonstrates how the spatial phase of the interference pattern is in second quantization encoded in the entanglement of the number states of the two halves of the trap.

Let us now relax the simplistic assumptions of our development. We allow for three-dimensional shapes of the atom trap and of the barrier that splits it. Moreover, we take into account the interactions between the atoms. These are characterized by the s-wave scattering length a and the corresponding interaction energy parameter $U_0 = 4\pi\hbar^2 a/m$, so that the two-body interaction is written $v(\mathbf{r}_i, \mathbf{r}_j) = U_0\,\delta(\mathbf{r}_i - \mathbf{r}_j)$. Inasmuch as a condensate can reasonably be represented with a Hartree type ansatz and the Gross-Pitayevski equation (GPE) [22–25], most of our analysis continues to hold true with straightforward modifications. The main difference is that one employs the solutions of the respective GPE for the wave functions ψ_0, χ_0, and $\chi_0^{\pm}$. The barrier should again be erected adiabatically, slowly compared to the inverse of the excitation frequencies of the trap, and it should be made high enough to render the tunneling time between the halves much longer than the time scale of the experiment. We assume that the interactions between the atoms do not significantly alter the entanglement from the form of Eq. (2). At the moment we have no proof to this effect, but at least in the limit of weak interactions the assumption clearly is valid.

With interactions between the atoms, phase diffusion becomes an issue. We assume that the barrier is erected fast enough so that phase diffusion does not intervene, and also ignore phase diffusion during the time between the release and the detection of the atoms. However, this time around we turn our scheme into a controlled experiment on phase diffusion: Before the atoms are released, they stay in the split trap a time t. The spatial phase of the observed interference pattern then varies at random from one experiment to the other. Our objective is to analyze how the statistics of the phase depends on the wait time t.

To get a handle on time evolution, we need the energies of the states $|k, N-k\rangle$. We reason as follows: For a given number k of atoms on, say, the left side, we first solve the GPE to obtain the self-consistent energy eigenstate χ_0^-. Since the GPE is the Hartree equation for the boson problem, the many-body state $\chi_0^-(\mathbf{r}_1)\chi_0^-(\mathbf{r}_2)\cdots\chi_0^-(\mathbf{r}_k)$ is a *variational minimum,* the product state with the lowest possible energy. Denoting the expectation value of the Hamiltonian in this state by $E_-(k)$, and similarly $E_+(k)$ for the right side, we assign the energy $E_-(k) + E_+(N-k)$ to the state $|k, N-k\rangle$.

For convenience we henceforth take the number of atoms N to be large and even, and continue to assume that both halves of the trap are identical, $E_-(k) = E_+(k) \equiv E(k)$. Since the atom statistics from Eq. (2) is binomial and strongly peaked around $k = N/2$, we expand the energy as

$$E_-(k) + E_+(N-k) \simeq \hbar[\omega_0 + \xi(k - N/2)^2], \quad (5)$$

with

$$\omega_0 = \frac{2E(N/2)}{\hbar}, \qquad \xi = \frac{1}{\hbar}\frac{d^2E(k)}{dk^2}\bigg|_{k=N/2}. \quad (6)$$

The state vector (2) evolves as

$$|\chi, t\rangle = \frac{e^{-i\omega_0 t}}{2^{N/2}}\sum_{k=0}^{N}\sqrt{\binom{N}{k}}\, e^{-i\xi t(k-N/2)^2}|k, N-k\rangle. \quad (7)$$

This shows scrambling of the phases, hence possible phase diffusion.

In earlier work models have been constructed that show how the act of measurement brings about a value for the condensate phase, even if the state of the condensate *per se* does not have any [7,9,13–16]. Here we do not go into such constructs. Instead, we propose a general framework for discussing measurements of condensate phase: Whatever the precise procedure of the experiments is, to a useful approximation the relative phase operator of the states $\chi_0^\pm$ as discussed in quantum optics literature by Luis and Sanchez-Soto [26] is measured [27]. The possible eigenvalues of the phase operator may be chosen as $\phi_p = 2\pi p/(N+1)$, $p = -N/2, -N/2+1, \ldots, N/2$, and the corresponding eigenstates are

$$|\phi_p\rangle = \frac{1}{\sqrt{N+1}}\sum_{k=0}^{N} e^{ik\phi_p}|k, N-k\rangle. \quad (8)$$

The orthonormal states $|\phi_p\rangle$, $p = -N/2, \ldots, N/2$, span the same Hilbert space as the states $|k, N-k\rangle$, $k = 0, \ldots, N$. Second, we will shortly see that the state (2), which corresponds to a fixed interference pattern, yields a distribution of phases peaked around 0 and having the narrow width $1/\sqrt{N}$. Third, whereas the phase φ in Eq. (4) was already noted to correspond to a φ shift of the phase of the spatial interference pattern of the two halves of the condensate, it is easy to see that the phase distribution predicted on the basis of Eq. (8) shifts by φ as well. These remarks provide an operational justification for the use of the states (8) as eigenstates of the relative phase between two condensates, given a fixed total atom number N.

For $N \gg 1$ we may use an integral to calculate the pertinent sum, and find

$$P(\phi) = |\langle\phi|\chi, t\rangle|^2 = \sqrt{\frac{\pi}{2(\Delta\phi)^2}}\exp\left[-\frac{\phi^2}{2(\Delta\phi)^2}\right], \quad (9)$$

where the root-mean-square width of the phase distribution is

$$\Delta\phi(t) = \sqrt{\frac{1}{N} + N\xi^2 t^2}. \quad (10)$$

Here we treat the phase ϕ as a continuous variable, and adjust the normalization so that the integral of the phase distribution is unity. At $t = 0$ the width of the phase distribution is $1/\sqrt{N}$. It is unclear how much stock should be put on the initial width, because the precise procedure for measuring the phase may well affect the results on the $1/\sqrt{N}$ level. The spreading of the phase distribution at the rate $R = \sqrt{N}\xi$ at later times probably is not as sensitive to such a caveat. As the width of the phase distribution grows linearly in time [17] and not as $\sqrt{t}$, the term "diffusion" may not be the most appropriate one. In analogy with the spreading of a Gaussian wave packet, "dispersion" might be more accurate. However, we continue to comply with the entrenched terminology.

To get a feel for the numbers, we take both sides of the trap to be harmonic oscillator wells with the geometric mean of the three trapping frequencies and the corresponding length scale given by ω and $\ell = \sqrt{\hbar/m\omega}$. Solving the GPE and calculating the energy $E(k)$ within the Thomas-Fermi approximation [22,23], i.e., ignoring kinetic energy altogether, we find

$$R = \left(\frac{72}{125}\right)^{1/5}\left(\frac{a}{\ell}\right)^{2/5}\frac{\omega}{N^{1/10}}. \quad (11)$$

Although the corresponding results in Refs. [18] and [21] display the same functional dependence on the parameters, they differ from Eq. (11) in the numerical factors. One reason is that these authors in effect write $E(k) = k\mu(k)$, where $\mu(k)$ is the chemical potential obtained from the GPE, while our method amounts to setting $E(k) = \int dk\, \mu(k)$. The latter form may be derived directly from the variational principle underlying the GPE, and also concurs with the thermodynamics definition $\mu = \frac{\partial E}{\partial N}$. In current experiments a typical value for the ratio of scattering length to trap size parameter is $a/\ell \sim 10^{-3}$ and the number of condensate atoms is

$N \sim 10^6$, so $R \sim 0.01\,\omega$. The rate of phase diffusion is two orders of magnitude smaller than the excitation frequencies of the trap, implying a phase diffusion time of the order of one second.

According to Eq. (7) the initial state vector, and hence the initial phase distribution, is regenerated at integer multiples of the time $T = 2\pi/\xi = 2\pi\sqrt{N}/R$. Such revivals [18,21] are beyond the continuum approximation for large N, which we have used to arrive at Eq. (9).

We have carried out our analysis without ever invoking spontaneously broken gauge symmetry. If one were to adhere to symmetry breaking, one should also postulate what happens to the phase when the condensate is cut in two. Admittedly such a supplementary rule is easy to come by. One begins with the initial condensate having a phase assigned by virtue of symmetry breaking, and integrates the GPE in time as the barrier is being raised. By symmetry, such an argument reproduces our prediction that the phases of the two halves come out the same. Nevertheless, spontaneous breaking of gauge symmetry is a postulate above and beyond statistical mechanics, and the statement that the evolution of the phase is described by the GPE is another. In fact, if one takes the latter assumption literally, the conclusion is that there never is any phase diffusion. More postulates evidently have to be added if one desires to study phase diffusion within the framework of symmetry breaking. When one starts down the road of making up a new principle, chances are that additional principles are also needed—and eventually it becomes hard to know which are the right ones. This is the essence of our objection against spontaneous symmetry breaking as the concept is introduced in statistical mechanics and condensed-matter physics: it does not lead to unambiguous answers to all legitimate experimental questions.

Of course, we have ourselves assumed that the relative-phase operator of Ref. [26] is useful as a generic description of a measurement of phase. We promote this approach for reasons of expediency. Nevertheless, the relative-phase operator could (in principle) be verified or disproved with an *ab initio* analysis of any given experiment for measuring the phase. While we have not yet carried out such studies, we expect them to be illuminating.

In summary, we have provided a theoretical description of an experiment in which a Bose-Einstein condensate is split in two, and the parts are then allowed to interfere. We show that studies of the interference pattern with different time delays between splitting and atom detection amount to quantitative tests of the recently developed notions of condensate phase [7,9,13–16] and phase diffusion [17–21]. Only the insufficient mechanical stability of the apparatus [3] now seems to stand in the way of the experiments.

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