Damping of the Transverse Head-Tail Instability by Periodic Modulation of the Chromaticity

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An analytical and numerical study of the suppression of the transverse head-tail instability by modulating the chromaticity over a synchrotron period is presented. We find that a threshold can be developed, and it can be increased to a value larger than the strong head-tail instability threshold. The stability criterion derived agrees very well with the simulations. The underlying physical mechanisms of the damping scheme are rotation of the head-tail phase such that the instability does not occur, and Landau damping due to the incoherent betatron tune spread generated by the varying chromaticity. [S0031-9007(97)03326-7]

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A bunched beam traveling in a particle accelerator creates forces through interactions of the beam particles with the electromagnetic environment. These forces, the so-called wakefields, react and perturb the beam, often causing collective instabilities. These instabilities limit the peak current in the bunch. In this Letter, we analyze a new method for controlling such instabilities, namely, through a temporal variation of the ring parameters. We illustrate this method with a practical example, the suppression of the transverse head-tail (HT) instability by means of variation of the chromaticity.

In a storage ring, particles with a different momentum have a different focusing strength in the quadrupoles, and thus have a different betatron frequency. The ratio of the relative frequency difference to the relative momentum difference is the chromaticity. The betatron angular frequency of an off-momentum particle is given by $\omega_\beta(\delta) = \omega_{\beta 0}(1 + \xi\delta)$, where ξ is the chromaticity, $\omega_{\beta 0}$ is the betatron angular frequency of the on-momentum particle, and $\delta = \triangle p/p$ is the relative momentum difference. The bunch is maintained by rf fields, and the particle energy oscillates with the synchrotron period. When $\xi = 0$, and a threshold depends on bunch current and wake force is exceeded, the strong head-tail (SHT) instability occurs. When $\xi \neq 0$, there are both the SHT instability with a threshold and the HT instability. The HT instability, driven by the chromaticity, has no stability threshold. It was observed in experiments [1], has been well analyzed [2], and has been confirmed by simulations [3].

The HT instability is a concern for many circular accelerators; for example, we may note the observations and simulations of single-bunch transverse excitation of the beam in the proton ring of the HERA collider at DESY [4], the observation of higher-order HT instability in the PS Booster of the LHC at CERN [5], and the investigation of the possible HT oscillation due to a transverse feedback kicker at KEK's B-Factory (KEKB) [6].

For most accelerators, ξ must be sufficiently small so as to avoid single particle orbital resonances. Analysis [7] shows that, under this circumstance, the growth rate of the HT instability is smaller (approximately by a factor of 3 for a typical case) when $\xi/\eta > 0$ than when $\xi/\eta < 0$, where $\eta = pdC/Cdp - 1/\gamma^2$ is the slippage factor, $C = 2\pi R = cT_0$ is the circumference of the ring, $\gamma = (1 - \beta^2)^{-1/2}$, and we take $\beta = v/c \approx 1$. The growth rate of the instbility depends on the magnitude of ξ/η and how the beam spectrum overlaps with the impedance (Fourier transform of the wakefield) spectrum. Damping mechanisms, such as radiation damping and Landau damping [7,8], are often used to stabilize the fast growing mode associated with the HT instability. Their effectiveness depends on the damping time and the width of the incoherent tune spread.

A moment expansion of the linearized Vlasov equation describing the coupled longitudinal and dipole transverse motion results in a lowest order mode with a large growth rate for $\xi/\eta < 0$, and strong damping when $\xi/\eta > 0$. Higher order modes grow, with substantially smaller growth rates than the lowest order mode, when $\xi/\eta > 0$. They are damped for $\xi/\eta < 0$.

This leads us to consider variation of ξ over the synchrotron period as a mechanism to suppress the HT instability. Two effects can be anticipated, first an enhanced Landau damping from the incoherent tune spread induced by the chromaticity variation, and, second, a strong focusinglike effect (on collective modes, instead of on single particle orbits) if the sign of ξ/η is changed.

We choose to vary ξ rather than η, since varying η means transition crossing, and thus involves many complicated phenomena, such as vanishing Landau damping, large momentum spread, bunch-shape mismatch, and nonlinear effects [9].

While drafting this Letter, we were advised of the existence of the paper written by Nakamura of SPring-8 [10]. Nakamura suggested, as we have also (independently), the concept of chromaticity modulation. In this Letter, going considerably beyond what Nakamura has done, we provide analysis, simulation results, and a stability criterion for the head-tail instability.

The chromaticity is assumed to vary as

$$\xi(s) = \xi_0 + \xi_1 \sin\phi\,, \tag{1}$$

which is a function of "time" s, where s measures the distance around the ring, $\phi = \omega_s s/c$, ω_s is the synchrotron angular frequency, and ξ_0 is the constant (dc) chromaticity. The constant part of chromaticity causes the HT instability. The dc incoherent tune spread is not effective in stabilizing the HT instability. As will be shown, the varying component of the chromaticity does not cause instability, but rather provides an incoherent tune spread that suppresses the coherent instability. This incoherent chromatic tune spread can be estimated as

$$\sigma_\nu = \nu_{\beta 0}\xi_1\sqrt{\langle \delta^2 \sin^2\phi\rangle} = \sqrt{3/8}\,\nu_{\beta 0}\xi_1\sigma_\delta\,, \tag{2}$$

for a Gaussian beam, where $\nu_{\beta 0} = \omega_{\beta 0}/\omega_0$, $\omega_0 = c/R$, $\sigma_\delta = (\omega_s/c\eta)\sigma_z$, σ_z is the rms bunch length,

$$z = r_z\cos\phi\,, \qquad \delta = (\omega_s/c\eta)r_z\sin\phi\,, \tag{3}$$

(r_z, ϕ) are the action-angle variables in the longitudinal phase space, and the bracket $\langle\rangle$ means a longitudinal phase-space average. The ac incoherent chromatic tune spread contributes to Landau damping and decoherence. Decoherence is an effect that causes decay of centroid oscillation of an off-centered beam with frequency spread, and is an excitation response to a nonzero initial condition [7,8]. The decoherence rate per turn can be estimated as $\tau_{\rm dec}^{-1} \approx 2\pi\sigma_\nu$.

The longitudinal motion is prescribed by Eq. (3). The transverse equation of motion is

$$\frac{d^2}{ds^2}y(z,s) + \frac{\omega_\beta^2(\delta)}{c^2}y(z,s) = -\frac{r_0}{\gamma C}\int_z^\infty dz'\,\rho(z')W(z-z')y(z',s)\,, \tag{4}$$

where $y(z)$ is the transverse (longitudinal) oscillation coordinate with respect to the bunch center, $N = \int dz'\,\rho(z')$ is the number of particles in a bunch, $\rho(z)$ is the beam density distribution, $r_0 = e^2/m_0c^2$, and W is the transverse wake function. We have neglected longitudinal wake force, nonlinear slippage factor, and synchrobetatron coupling. There are three parameters essential to the dynamics given by Eqs. (3) and (4):

$$\chi_0 = \omega_{\beta 0}\xi_0\sigma_z/c\eta\,, \qquad \chi_1 = \omega_{\beta 0}\xi_1\sigma_z/c\eta\,, \tag{5}$$

$$\Upsilon \equiv \pi N r_0\langle W\rangle c^2/8\gamma C\omega_{\beta 0}\omega_s\,, \tag{6}$$

where χ_0 (χ_1) is the dc (ac) phase shift between the head and tail of a bunch, and $\langle W\rangle = \int_{-\infty}^{\infty} dz'\,\rho(z')W(z-z')$. The parameter Υ is about the ratio of betatron tune shift to the synchrotron tune. When $\chi_1 = 0$, the SHT instability occurs when $\Upsilon \geq 1$ [7,11].

The nonlinear part of the chromaticity (as contrasted with the varying chromaticity), characterized by ξ_{01}, also generates an incoherent tune spread, where $\xi_{\rm dc} = \xi_0 + \xi_{01}\delta$. This tune spread is smaller by a factor of σ_δ than the tune spread from the varying chromaticity, and is not significant in most instances.

To develop an analytic criterion for the stability threshold, we have derived a dispersion relation based on a linearized Vlasov analysis for a many-particle system. As a starting point of the analysis, we calculate the HT growth rate for arbitrary ξ_0 and ξ_1, neglecting any damping from tune spread. We write the perturbed longitudinal phase space distribution function in action-angle variables and then Fourier expand in modes that vary as $e^{-il\phi}$, where ϕ is the angle variable and l is the mode index. The result [12] is a complex mode frequency, $\Omega^{(l)}$, approximately given by

$$\Omega^{(l)} = \omega_{\beta 0} + l\omega_s + \omega_0\Delta\nu\,. \tag{7}$$

We assume the beam distribution is Gaussian, and take a model-impedance function as $\tilde{Z}(\omega) = 1/\omega - i\pi\delta(\omega)$, corresponding to a uniform wake function. The coherent tune shift is

$$2\pi\Re(\Delta\nu) \approx -(4\Upsilon/l!\,2^l)\nu_s\chi_0^{2l}e^{-\chi_0^2}J_0^2(\chi_1/4)\,, \tag{8}$$

where $\nu_s = \omega_s/\omega_0$ and $J_0(x)$ is the Bessel function. The growth rates per synchrotron period, $1/\tau_s^{(l)} = 2\pi\Im(\Delta\nu)/\nu_s$, of the two lowest order modes are

$$1/\tau_s^{(0)} \approx -4\Upsilon\,\mathrm{Erfi}\,(\chi_0)e^{-\chi_0^2}J_0^2(\chi_1/4)\,, \tag{9}$$

$$1/\tau_s^{(1)} \approx \sqrt{\pi}\,\Upsilon\chi_0 L_{1/2}^{(-1/2)}(\chi_0^2)e^{-\chi_0^2}J_0^2(\chi_1/4)\,, \tag{10}$$

where $\mathrm{Erfi}\,(x) = -i\mathrm{Erf}\,(ix)$, $\mathrm{Erf}\,(x)$ is the error function, and $L_k^{(l)}(x)$ is the Laguerre polynomial. One can see that, when $\chi_0 = 0$, the growth rate of the HT instability is zero, even when $\chi_1 \neq 0$. As mentioned, the ac part of the incoherent tune spread contributes to a Landau damping without driving the HT instability.

We can estimate the stability criterion by requiring that the incoherent tune spread exceeds the absolute value of the coherent tune shift. That is, $\sigma_\nu > [\Delta\nu]$, where σ_ν is given by Eq. (2), or

$$\chi_1 > (8/\pi)\sqrt{2/3}\,N_l\Upsilon|\tilde{Z}_{\rm eff}^{(l)}(\chi_0)|\,, \tag{11}$$

where $N_l = \int d\omega_p|g_l|^2$, g_l is the frequency spectrum of the lth mode of the perturbed beam density, the transverse impedance is $Z(\omega_q) = -W\tilde{Z}(\omega_q)$, the effective impedance is

$$\tilde{Z}_{\rm eff}^{(l)} = \frac{\sum_q \tilde{Z}(\omega_q)\,|g_l(\chi_1, \chi_q - \chi_0)|^2}{\sum_q |g_l(\chi_1, \chi_q - \chi_0)|^2}\,, \tag{12}$$

$\chi_q = \omega_q\sigma_z/c$, and $\omega_q = q\omega_0 + \omega_{\beta 0} + l\omega_s$. Expressed in terms of the accelerator parameters, we have

$$\xi_1 > c_l\frac{eI_0|Z_1^{\perp(l)}(\xi_0)|_{\rm eff}}{E}\left(\frac{R}{\sigma_z}\right)^2\left(\frac{\eta R}{\nu_s\nu_{\beta 0}^2}\right), \tag{13}$$

where $c_l = \sqrt{2/3}\,\Gamma(l + 1/2)/\pi l!\,2^{l+1}$, $E = \gamma m_0 c^2$, and $I_0 = Nec/C$ which is the averaged current. When $0 < \chi_0 < 1$, the $l = 1$ mode is usually the dominant unstable mode, and $c_1 = 0.058$; when $-1 < \chi_0 < 0$, the $l = 0$ mode is the dominant unstable mode, and $c_0 = 0.23$.

Equation (11) is usually sufficient for estimating the threshold for bunch centroid motions. An improved stability criterion, useful for estimating the threshold for a growth of the emittance, can be derived by incorporating the incoherent tune spread in the Vlasov analysis. Following the well-known techniques [13], one can find the dispersion relation,

$$V + iU = \frac{8\Upsilon}{2\pi} N_l \{\Re[\tilde{Z}_{\mathrm{eff}}^{(l)}] + i\Im[\tilde{Z}_{\mathrm{eff}}^{(l)}]\}, \qquad (14)$$

where $V + iU$ is the beam transfer function [7]. For a Gaussian beam with the model impedance and $\nu = \Delta\nu/\nu_s$, we have, for the $l = 0$ mode,

$$V + iU = \frac{-i\chi_1^2/2}{\sqrt{2\pi}\,\chi_1 - 2\pi\nu e^{-2\nu^2/\chi_1^2}[\mathrm{Erfi}(\frac{\sqrt{2}\nu}{\chi_1}) - i]}, \qquad (15)$$

and for the $l = 1$ mode,

$$V + iU = \frac{-i\chi_1^4}{\sqrt{2\pi}\,(\chi_1^3 + 4\nu^2\chi_1) - 8\pi\nu^3 e^{-2\nu^2/\chi_1^2}[\mathrm{Erfi}(\frac{\sqrt{2}\nu}{\chi_1}) - i]}. \qquad (16)$$

Examination of the dispersion relation shows that the SHT threshold can be enlarged by increasing χ_1.

A simulation code has been developed, which follows the motion of macroparticles that are initially loaded with a bi-Gaussian distribution in both longitudinal and transverse phase spaces. The motion of each particle is determined by Eqs. (3) and (4). Results are numerically converged when the number of macroparticles is larger than 400. Since χ_0 is usually chosen as a positive parameter, we show figures only for numerical work for $\chi_0 > 0$. Simulations, nevertheless, confirm the growth rates and stability criterion for both signs of χ_0. In Fig. 1, we plot the rms displacement of the bunch centroid averaged over a synchrotron period. For a beam with an initial centroid offset, the bunch centroid motion is initially dominated by the $l = 0$ mode, which is a damping mode when $\chi_0 > 0$; the higher order unstable modes then cause the growth of averaged bunch center after the initial damping. The varying chromaticity, nonetheless, Landau damps all the higher order unstable modes when χ_1 is larger than that estimated in Eq. (11).

Multiparticle simulations show that the rms emittance of a Gaussian beam is stabilized when the value of χ_1 approaches the stability threshold of Eq. (14) (cf. Fig. 2), where $\varepsilon_{\mathrm{rms}} = (\langle y^2\rangle\langle P_y^2\rangle - \langle yP_y\rangle^2)^{1/2}$ (cm), $P_y = (c/\omega_{\beta 0})dy/ds$ (cm), and the bracket $\langle\,\rangle$ means a phase-space ensemble average. Note that the emittance growth is much slower than the initial centroid damping [cf. Figs. 1 and 2]. This is a result of the growth rates of the unstable higher order modes ($l \geq 1$) being much smaller than the damping rate of the ($l = 0$) mode, e.g., $\tau_s^{-1}(l = 0) \approx -4\tau_s^{-1}(l = 1)$. Figure 3 shows that the results of simulation of the bunch centroid motion agree very well with the approximate stability limits, and the results of emittance growth agree with the exact stability criterion. In Figs. 1, 2, and 3, $\Upsilon = 0.22$; for other values of Υ, simulations also agree with the theoretical stability criterion (for $\Upsilon < 0.2$ such that the SHT effect is not prominent). Figures 4(a) and 4(b) show the stabilization of the SHT instability by a large enough χ_1. This implies that the limitation of peak current in a storage ring, within the tolerance of dynamic aperture reduction, can be increased by varying the chromaticity.

In summary, the chromaticity of a storage ring, which causes the head-tail instability, usually needs to be controlled by sextupoles. We have shown that, by varying the chromaticity, the head-tail instability is suppressed, and, futhermore, a stability threshold is developed. With

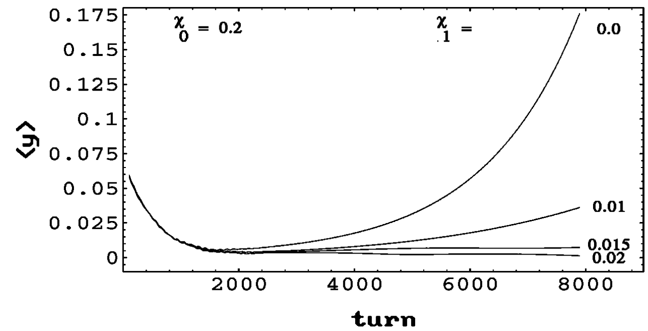

FIG. 1. Multiparticle simulation results showing the stabilization of the HT motions of the centroid of a Gaussian beam by χ_1, where $\chi_0 = 0.2$, $\Upsilon = 0.22$, $\nu_s = 0.0094$, $\langle y\rangle(0) = 0.1$ (cm), and $\varepsilon_{\mathrm{rms}}(0) = 0.01$ (cm). The estimated stability threshold for the $l = 1$ mode, according to Eq. (11), is $\chi_1 \geq 0.0127$.

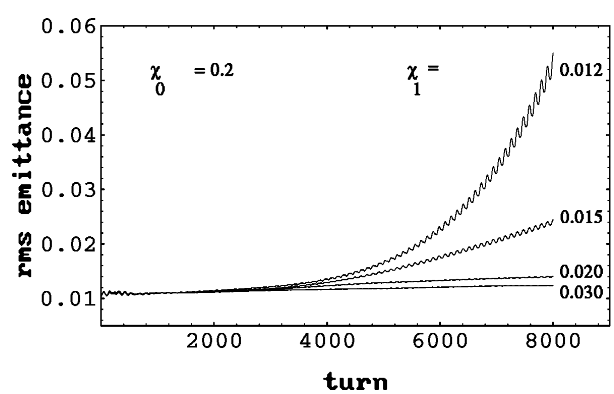

FIG. 2. Multiparticle simulation result showing the stabilization of the HT motions of the rms emittance of a Gaussian beam when $\chi_1 \geq 0.026$—the theoretical stability threshold of the $l = 1$ mode [cf. Eq. (14)]. Here $\chi_0 = 0.2$, $\Upsilon = 0.22$, $\nu_s = 0.0094$, $\langle y\rangle(0) = 0.1$ (cm), and $\varepsilon_{\mathrm{rms}}(0) = 0.01$ (cm).

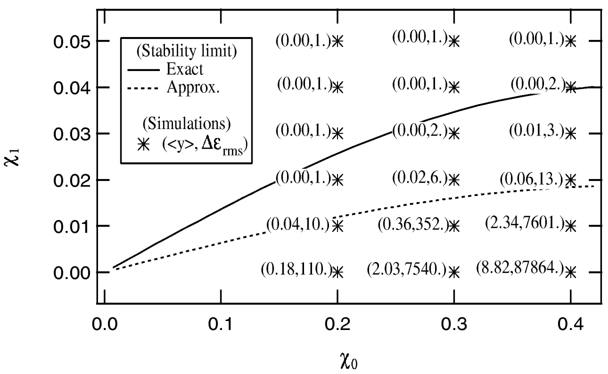

FIG. 3. Stability limits of a Gaussian beam with the model-impedance function, in the χ_1-χ_0 coordinate. Here $\Upsilon = 0.22$, $\langle y \rangle$ is the averaged centroid motion at 8000 turns, $\Delta\varepsilon_{\rm rms} = \varepsilon_{\rm rms}(8000)/\varepsilon_{\rm rms}(0)$, and the approximate and exact stable limits are plotted according to the criteria shown in Eqs. (11) and (14), respectively. The region above the solid (dashed) line is stable for the bunch's rms-emittance (centroid) motion. Here, $\nu_s = 0.0094$, $\langle y \rangle(0) = 0.1$ (cm), $\varepsilon_{\rm rms}(0) = 0.01$ (cm), and $\Delta\varepsilon_{\rm rms}$ is rounded to the closest integer.

a large enough allowable ac part of the chromaticity, one may increase the threshold of the strong head-tail instability. The underlying mechanisms are Landau damping and rotation of the head-tail phase. Studies of practical operation issues, such as rapidly modulated sextupole magnets, and theoretical issues, such as the reduction of dynamic aperture due to resonances, as well as exact calculations including the azimuthal mode coupling, are required. Also, practical aspects of the varying chromaticity must be compared with the other schemes that also introduce an incoherent tune spread, e.g., space charge, ion trapping, rf nonlinearity, and octupole magnets. Temporal variation of accelerator parameters might be used in the control of other instabilities.

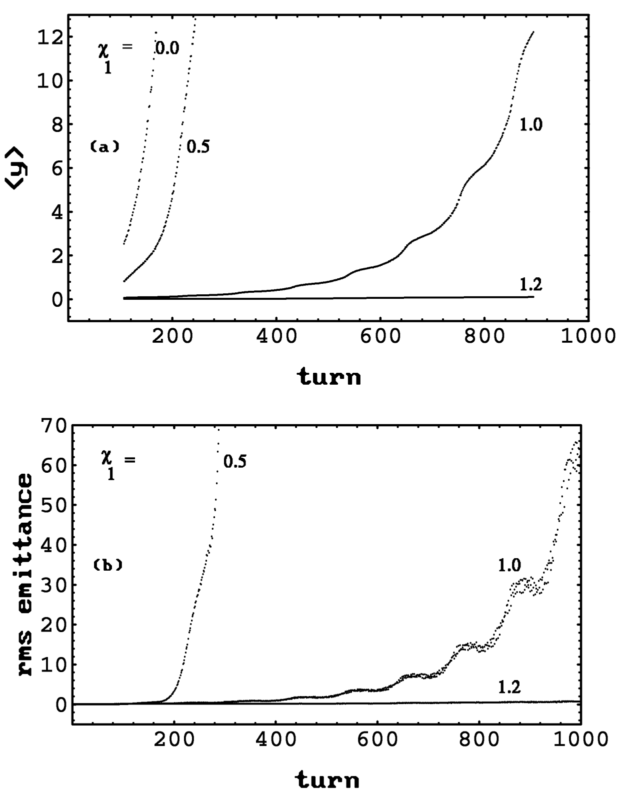

FIG. 4. Multiparticle simulation results showing stabilization of the SHT motions of (a) the centroid and (b) the rms emittance of a Gaussian beam by χ_1. The SHT stability limit is $\Upsilon < 1$, when $\chi_1 = 0$. In these figures, $\chi_0 = 0$, $\Upsilon = 1.65$, $\nu_s = 0.0094$, $\langle y \rangle(0) = 0.1$ (cm), and $\varepsilon_{\rm rms}(0) = 0.01$ (cm).

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