From High Dimensional Chaos to Stable Periodic Orbits: The Structure of Parameter Space

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Regions in the parameter space of chaotic systems that correspond to stable behavior are often referred to as windows. In this Letter, we elucidate the occurrence of such regions in higher dimensional chaotic systems. We describe the fundamental structure of these windows, and also indicate under what circumstances one can expect to find them. These results are applicable to systems that exhibit several positive Lyapunov exponents, and are of importance to both the theoretical and the experimental understanding of dynamical systems. [S0031-9007(97)03367-X]

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A characteristic feature of one dimensional chaotic dynamical systems is the appearance of stable behavior as system parameters traverse chaotic regions. For example, in the bifurcation diagram of the quadratic map $x \to x^2 - a$, large areas of chaotic behavior are visible, but are punctuated by parameter intervals in which stable periodic behavior is observed. These intervals, commonly called *windows*, have long been believed to be present arbitrarily close to every parameter value that leads to chaos. Only recently has this been proven to be true [1].

In this Letter, we address the fundamental problem of the occurrence of stable periodic behavior amid high dimensional chaos. We propose a conjecture that describes the nature of parameter space for chaotic maps, and, furthermore, indicates under what circumstances one may reasonably expect to have numerous parameter space regions that lead to stable periodic behavior (i.e., windows). This conjecture can be of considerable practical importance for experimentalists, since it is often desirable to establish nonchaotic behavior in the vicinity of parameter values that give rise to chaos.

We begin by describing the content of our conjecture in practical terms. We then motivate the work, and conclude with a precise mathematical statement of our result. Most chaotic systems discussed in the scientific literature are almost certainly "fragile" in the sense that a slight alteration of a large number N of parameters will destroy the chaos and replace it by a stable periodic orbit. Let k be the number of positive Lyapunov exponents of a chaotic attractor, but suppose that only $n < N$ parameters can be varied in an experiment. We conjecture that if $n \geq k$, then typically a slight change applied to these n parameters can destroy the chaos. If, however, $n < k$, then the chaos typically cannot be so destroyed. In this case, we expect that for an experimentally significant parameter space region near the original setting, the chaotic attractor will persist.

For example, if $k = 1$, then as one parameter is slightly varied, numerous stable regions will be observed. If $k = 2$, then slight changes to a single parameter will typically not destroy the chaos. However, if two parameters are available, then the parameter space can be systematically searched in two dimensions, and many windows can be located.

Knowledge of these windows may be helpful in controlling the system, even in the presence of noise. Alternatively, if the location of a desired window is to be calculated, our conjecture indicates that one must typically solve for at least $n = k$ parameters.

We now motivate the work. Our conjecture is based on the idea that a window is constructed around a *spine locus.* For simplicity, we consider maps that contain critical points [2]. For one dimensional maps, the spine locus corresponds to parameter values that give rise to superstable orbits. To illustrate, consider a map $x \to F(x; a)$, where a is a scalar parameter. The stability of a period p orbit is governed by $m = \frac{d}{dx} F^p(x) = \frac{d}{dx} F(x_p) \frac{d}{dx} F(x_{p-1}) \cdots \frac{d}{dx} F(x_1)$, where the derivatives are evaluated at each point in the orbit. The orbit is asymptotically stable if $|m| < 1$, and an orbit that contains a critical point of F, where $dF/dx = 0$, has $m = 0$ and is called a superstable orbit. As the parameter varies in the vicinity of the spine, m sweeps through the interval $(-1, 1)$. In this way, the extent of the window is delineated. For the quadratic family $x \to x^2 - a$, the windows are intervals in the (one dimensional) parameter space built around isolated spine points.

For maps with more parameters, bifurcation diagrams are usually drawn entirely in parameter space, with points shaded differently to represent the type of dynamics generated. In the case of the two parameter quadratic family $x \to (x^2 - a)^2 - b$, the spine locus consists of two parabolas; see Fig. 1. The black curves, defined by the condition $m = 0$, are the spine locus; these clearly determine the shape of the window.

Of importance for our purposes is the dimension of the spine locus. In particular, we note that the condition $m = 0$ is a single constraint, and hence the spine locus is of codimension one in the parameter space (i.e., one less than the parameter space dimension).

The dimension of the spine determines the geometry of the window in the following sense. If the spine is a

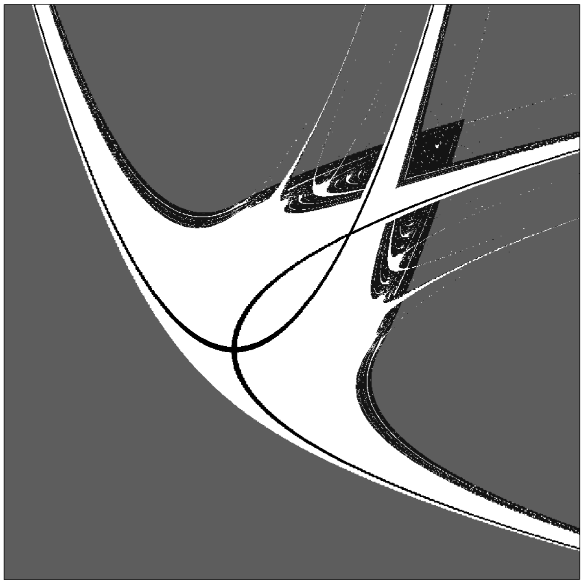


FIG. 1. The bifurcation diagram for $x \rightarrow (x^2 - a)^2 - b$. The axes represent $a, b \in [-2, 3]$. White areas lead to stable periodic orbits, while light grey points lead to divergent trajectories. Dark grey points give rise to chaos with one positive Lyapunov exponent. Superimposed in black is the spine locus, which delineates the shape of the window.

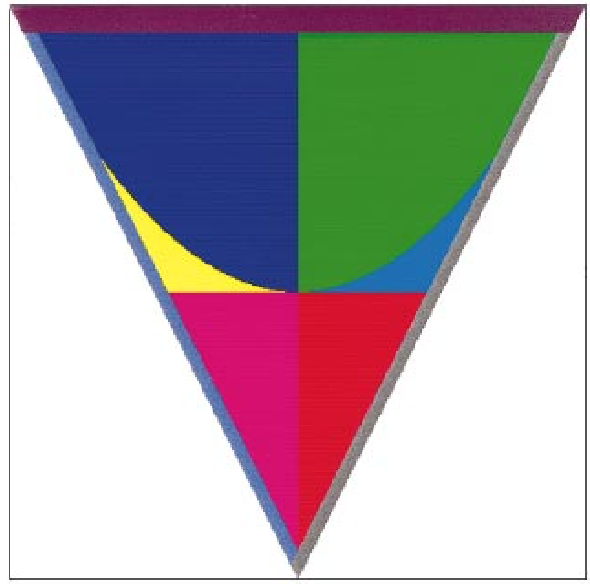


FIG. 2(color). The stability triangle. The trace $T \in [-2, 2]$ is graphed horizontally, and the determinant $D \in [-1, 1]$ is graphed vertically.

point, then the window will typically have limited extent. We call such windows *limited.* This is to be contrasted with windows that have spines of higher dimension. In this case, the window extends along the entire length of the spine, as in Fig. 1. We call such windows *extended.* (These notions are made more precise below.)

For two dimensional maps, the identification of the spine locus is more involved. Let $\mathbf{x} \rightarrow \mathbf{F}(\mathbf{x}; \mathbf{a})$, where $\mathbf{x}$ is a two dimensional state vector, and assume that there are n parameters so that $\mathbf{a} \in \Re^n$. A period p orbit is asymptotically stable if $|\lambda_i| < 1, i = 1, 2$ where the λ's are the eigenvalues of $\mathbf{M}$, the Jacobian matrix of the p-times iterated map: $\mathbf{M} = \mathbf{DF^p}(\mathbf{x}) = \mathbf{DF}(\mathbf{x}_p) \cdot \mathbf{DF}(\mathbf{x}_{p-1}) \cdot \cdots \cdot \mathbf{DF}(\mathbf{x}_1)$.

First, consider a region of parameter space that exhibits only one positive Lyapunov exponent $h_1 > 0$, such that $h_1 + h_2 < 0$. Here, the map is asymptotically area contracting, and on average, $|\det(\mathbf{DF})| < 1$ over the course of a trajectory. For a periodic orbit, we have $D = \det[\mathbf{M}(\mathbf{x})] = \lambda_1 \lambda_2 \approx 0$ for sufficiently high p, and thus at least one eigenvalue is close to zero. The stability requirements therefore reduce to one condition for stability, and the spine loci in this region are of codimension one.

Now consider parameter regions that correspond to two positive Lyapunov exponents. For this case, it is advantageous to recast the stability conditions in terms of the trace $T = \lambda_1 + \lambda_2$ and determinant $D = \lambda_1 \lambda_2$ of $\mathbf{M}$. Stability implies that these numbers must fall within a triangular region in D versus T space, shown in Fig. 2. We refer to this region as the stability triangle. Every parameter space point that leads to a stable orbit maps to a particular point within the stability triangle.

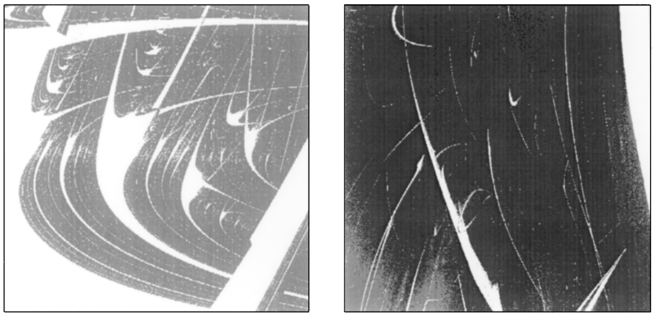
Of central importance is the point where $D = T = 0$. We refer to this point and the corresponding parameter space points as *nilpotent* points. The spine locus for windows in this case consists of nilpotent parameter values. By coloring points within a window according to their corresponding location in the stability triangle as in Fig. 2, the above construction makes nilpotent points in parameter space easily discernible as points where the colors come together. Note that the restriction of D and T to the stability triangle represents two constraints, and therefore the spine locus is of codimension two.

We illustrate these ideas with a two dimensional, two parameter map (see also [3]): $(x, y) \rightarrow (\alpha x(1 - x) + (1 - \frac{\alpha}{4})y, \beta y(1 - y) + (1 - \frac{\beta}{4})x)$. A physically motivated map, the kicked double rotor [4], has been observed to have a similar parameter space structure.

Figure 3(a) shows a region of parameter space dominated by area-contracting chaos with one positive Lyapunov exponent. The spines in this region are one dimensional, and we find very many extended windows.

Figure 3(b) shows a region of area-expanding chaos with one positive Lyapunov exponent. The spines are again one dimensional, and we see many extended windows. We note that in this region, the windows are qualitatively different than those in 3(a) [5]. Nevertheless, the windows are consistent with our conjecture.

Of primary interest for this Letter are regions where two positive Lyapunov exponents are found. Here the spines consist of isolated nilpotent points, and we find a large number of limited windows in the two dimensional parameter space. Figure 4(a) shows a section of this

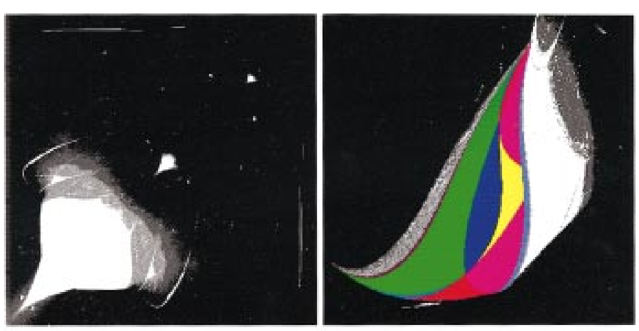


(a) (b)

FIG. 3. (a) A region in parameter space $(\alpha, \beta) \in [3.4722, 3.4857] \times [1.078, 1.316]$ dominated by area-contracting chaos with one positive Lyapunov exponent (light grey). White areas lead to asymptotically stable orbits. As predicted by our conjecture, a dense set of extended windows is seen. In (b) $(\alpha, \beta) \in [2.876, 3.288] \times [1.932, 2.46]$, and dark grey areas indicate area-expanding chaos with one positive Lyapunov exponent. Again, a dense set of extended windows is seen, as predicted.

region, and the windows indeed appear to be dense and limited. (The long, thin windows at the top and right of the figure are limited in extent, and qualitatively similar to other windows in the region.) Figure 4(b) shows a blowup of a period five window with the interior colored according to Fig. 2. It is immediately apparent that the window is constructed around the two isolated nilpotent points at the top and bottom of the central blue region. Other windows from this region are similarly constructed; some contain only one nilpotent point.

The identification of the spine locus can be expanded to d dimensional maps. In this case the matrix $\mathbf{M}$ is $d \times d$, and hence has a characteristic polynomial of degree d in λ. The coefficients c_i can be written as



(a) (b)

FIG. 4(color). (a) A parameter space region $(\alpha, \beta) \in [3, 4] \times [3, 4]$ dominated by chaos with two positive Lyapunov exponents (black). The shading is otherwise as in Fig. 3. A dense set of limited windows is seen, as predicted by our conjecture. In (b) a window from within (a) is magnified, $(\alpha, \beta) \in [3.375, 3.42] \times [2.87, 2.9825]$, and the interior of the period five region is colored according to Fig. 2. Two nilpotent points, forming the spine, are evident at the top and bottom of the central blue region.

the sum of all possible distinct product combinations of the eigenvalues taken i at a time, for $i = 1, 2, \ldots, d$. The stability requirements $|\lambda_i| < 1, i = 1, 2, \ldots, d$ then determine a volume in the coefficient space, and stability occurs if the numbers $c_1, c_2, \ldots, c_d$ lie within this volume. The spines of windows are given by points in parameter space that correspond to the center of this volume, where $c_1 = c_2 = \cdots = c_d = 0$. These equations may be solved numerically to locate particular windows.

As described above for the two dimensional case, some of the conditions restricting the parameter space may be automatically satisfied by the dynamics being considered. In an attractor on which all invariant measures yield k positive Lyapunov exponents, unstable periodic orbits have at most k expanding directions. Therefore, the transition to stability involves satisfying at most k requirements, and hence the spines of windows are of codimension k in the parameter space.

Knowledge of the dimension of the spine locus gives information about when one may expect to find windows. Assume that one has available n parameters. Varying these parameters defines an n dimensional accessible parameter manifold within the full parameter space. In order to observe windows, this accessible parameter manifold must intersect (or come close to) a spine locus for some period p. (For maps without critical points, e.g., the Hénon map, the determinant is bounded away from zero, but can nevertheless come very close to zero for high p.) If the codimension of the spine locus is k, then typically the accessible parameter manifold must be of dimension at least $n = k$ for point intersections to generically occur. In this case the windows, as viewed in the accessible parameter space, are constructed around isolated spine points, and therefore are limited. If the accessible parameter manifold is of a higher dimension, typical intersections occur in higher dimensional sets, and therefore we expect extended windows in the accessible parameter space. Finally, because unstable periodic orbits are dense in a chaotic attractor, we expect that arbitrarily small perturbations to k parameters can stabilize one of these orbits (as occurs in the one dimensional case). Thus we expect windows to be dense when $n \geq k$.

We now state our conjecture more precisely, beginning by introducing a few definitions that facilitate the presentation. Let f be a smooth map from a region $S \subset \Re^d$ to itself that exhibits a chaotic attractor Λ with k positive Lyapunov exponents (we assume for simplicity that all invariant measures supported on Λ yield the same k). Let g be a map close to f (by which we mean that $f(\mathbf{x})$ and $g(\mathbf{x})$ are close and that all first partial derivatives of g are close to those of f). We say that Λ is *dispelled* for g if almost all points in a neighborhood of Λ belong to basins of attracting periodic orbits of g. The situation is illustrated in Fig. 5 using the Hénon map $(x, y) \rightarrow (\rho - x^2 + (0.3)y, x)$. If there exist (possibly rare) functions arbitrarily close to f for which the attractor Λ is dispelled, we say that Λ is *fragile* [6]. Finally, consider an n-parameter family of

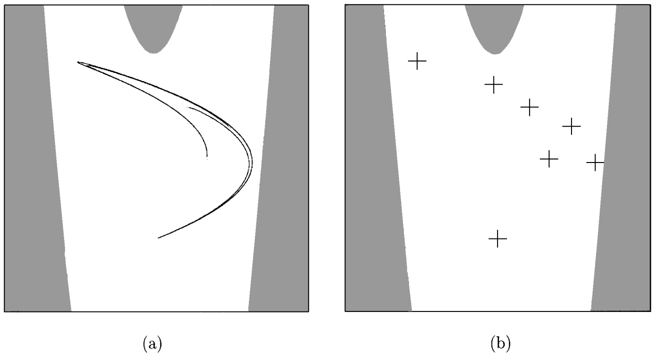


FIG. 5. Illustration of a *dispelled* chaotic attractor, using the Hénon map as in the text. The figures show $(x, y) \in [-2.5, 2.5] \times [-2.5, 2.5]$. In (a) $\rho = 1.22$ and the white region is the basin of the chaotic attractor shown in black. The grey region is the basin of infinity. In (b) $\rho = 1.23$ and trajectories originating in the vicinity of the attractor in (a) now converge to the attracting period seven orbit shown (crosses). We say that the chaotic attractor in (a) is dispelled for the map with $\rho = 1.23$.

functions $f_{\mathbf{a}}$, where $\mathbf{a} \in \Re^n$, such that $f_{\mathbf{0}} = f$ and $f_{\mathbf{a}}$ depends smoothly on $\mathbf{a}$. We define the *window set* W to be the set of $\mathbf{a}$ values such that Λ is dispelled for $f_{\mathbf{a}}$.

Windows conjecture.—Let f be a smooth map from a region $S \subset \Re^d$ to itself that exhibits a fragile chaotic attractor Λ with $k \geq 1$ positive Lyapunov exponents, where all invariant measures supported on Λ yield the same k. Let W be the window set corresponding to a typical family $f_{\mathbf{a}}$, where $\mathbf{a} \in \Re^n$ and $f_0 = f$. (1) If $n < k$, there exists a neighborhood of $\mathbf{a} = \mathbf{0}$ entirely outside of W. (2) If $n = k$, W is dense in a neighborhood of $\mathbf{a} = \mathbf{0}$ and the components of W are limited. (3) If $n > k$, W is dense in a neighborhood of $\mathbf{a} = \mathbf{0}$ and the components of W are extended.

The number d represents the dimension of the state vector, and n is the number of accessible parameters. We expect that in cases (2) and (3), W consists of a union of connected subsets w_i; these are the individual windows. By *limited* in case (2) we mean intuitively that the subsets w_i get smaller and smaller as they converge to $\mathbf{0}$. That is, as we look in successively smaller neighborhoods of $\mathbf{0}$, the diameters of the w_i decrease to zero [7]. In case (3), we expect that this property does not hold, and we call the components w_i *extended.* In this case, the w_i may be quite long in the vicinity of $\mathbf{0}$ (as in Fig. 1).

Our conjecture describes the local structure of parameter space in the vicinity of a point $\mathbf{a} = \mathbf{0}$ that gives rise to chaos. We believe, however, that it has important implications for the global structure as well. For the one dimensional quadratic family, it is known that the set of parameter values that give rise to chaos has a nonzero Lebesgue measure [8]. In this sense, chaos is common. We expect that in more general higher dimensional situations, chaos with several positive Lyapunov exponents is similarly common. If this is so, then by applying our conjecture at every such point, we can infer global properties of the parameter space. We leave a rigorous treatment of our results to future efforts in light of the extreme difficulty of the proof in Ref. [1].

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[1] J. Graczyk and G. Świątek, "Hyperbolicity in the Real Quadratic Family" Ann. Math. (to be published).
[2] Results derived from one dimensional canonical maps with critical points have been seen to carry over to more general cases such as the Hénon map. See, for example, J. A. Yorke, C. Grebogi, E. Ott, and L. Tedeschini-Lalli, Phys. Rev. Lett. **54**, 1095 (1984); B. R. Hunt, J. A. C. Gallas, C. Grebogi, J. A. Yorke, and H. Kocak (to be published).
[3] L. Gardini, R. Abraham, R. J. Record, and D. Fournier-Prunaret, Int. J. Bifurcation Chaos **4**, 145 (1994).
[4] E. J. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke, Physica (Amsterdam) **25D**, 347 (1987); Phys. Lett. A **118**, 448 (1986); **120**, 497(E) (1987).
[5] E. Barreto, B. Hunt, C. Grebogi, and J. A. Yorke (to be published).
[6] We believe that most chaotic systems described in the literature are fragile. We point out that the combined results of Refs. [1] and [8] prove that attractors for the quadratic family are fragile.
[7] The diameter of a set is the maximum distance between two points of the set.
[8] M. V. Jacobson, Commun. Math. Phys. **81**, 39 (1981).
[9] H. E. Nusse and J. A. Yorke, *Dynamics: Numerical Explorations* (Springer-Verlag, New York, Heidelberg, Berlin, 1994).