Sculpting of a Fractal River Basin

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The principle of reparametrization invariance is used to derive a dynamical equation for the erosion of the landscape of the drainage basin of river networks. The stationary solutions of the equation are found to have scaling behavior that is consistent with observational data. Our analytic prediction of the main stream profile is confirmed by numerical results and is amenable to direct observational verification. [S0031-9007(97)03339-5]

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Drainage basins of rivers evolve into striking fractal forms as a result of erosional processes [1]. Soil height maps [2] of such self-organized landscapes have been used to study scale-free (algebraic) distributions of several attributes of the rugged landscape. On a lattice, the transportation network in a river basin is a spanning tree that provides a unique route for water flow from each site (representing a unit area of the drainage basin) to the outlet. From each site, the local flow is downhill to the neighboring site with the lowest height. The landscape is characterized by variables z_i, a_i, l_i, and r_i at the ith site, representing the soil height, the accumulated area or the number of sites upstream of i that eventually drain into i, the upstream length or the distance measured along the stream to the farthest upstream site that drains into i, and the mean precipitation per unit time at site i, respectively.

Observations reveal a consistent correlation between the local gradient of the soil height and the accumulated area at that site [3]

$$\langle|\vec{\nabla}z|\rangle \propto a^{-1/2}, \tag{1}$$

where the average is over all sites draining an area a. Equation (1) is known as the slope-discharge relation. The distributions of a_i and l_i are characterized by power laws with exponents τ and γ, respectively. Also a_i and l_i are found to be correlated through the relationship $l \sim a^h$ (h is called the Hack exponent). River basins around the world are found to have values of τ, γ, and h in the range 1.41–1.45, 1.67–1.85, and 0.54–0.60, respectively. These exponents are not independent but they are related by [4,5]

$$\tau = 2 - h, \tag{2}$$

$$\gamma = \frac{1}{h}. \tag{3}$$

In this Letter, we derive an evolution equation from general considerations, the solution of which provides a quantitative explanation of the observed facts. We also predict the scaling of the main stream profile that may be deduced from observational data and would provide a test of our theory.

The evolution of the surface of a landscape under the effect of erosion [6,7] can be generally described by

$$\partial_t\vec{r}(\underline{p},t) = -\hat{n}(\underline{p},t)\mathcal{F}[\vec{r}(\underline{p},t), J(\vec{r}(\underline{p},t)), \vec{g}], \tag{4}$$

where $\vec{r}(\underline{p},t)$ is a three dimensional vector spanning the surface, $\underline{p} = (p_1, p_2)$ describes the parametrization of the surface, $\hat{n}(\underline{p},t)$ is the vector normal to the surface at $\vec{r}(\underline{p},t)$, $J(\vec{r}(\underline{p},t))$ is a measure of the water flow and simply proportional to the drained area a (J can be treated as a scalar because its direction is determined by steepest descent along the landscape), and $\vec{g}$ is the acceleration due to gravity. The time derivative of $\vec{r}$ must be parallel to $\hat{n}$ since a tangential component would merely lead to a change in the parametrization. Because no erosion takes place when $J = 0$ or when one has a completely flat landscape ($\hat{n}$ and $\vec{g}$ are antiparallel then) and because the equation must be invariant under reparametrization [8] and thence involve only intrinsic quantities, one obtains to lowest order

$$\mathcal{F} = \frac{2}{|\vec{g}|}\beta a[|\vec{g}| + \hat{n}\cdot\vec{g}]. \tag{5}$$

The constant coefficient β can be set equal to 1 on defining the time units appropriately. In the Monge parametrization, in which $\underline{x}$ is a two dimensional vector in the "substrate plane" and $z(\underline{x})$ is the height of the surface in the z direction perpendicular to the plane, the equation becomes

$$\dot{z} = -2a\left[\sqrt{1 + |\vec{\nabla}z|^2} - 1\right], \tag{6}$$

which in a small height gradient approximation yields

$$\dot{z}(\underline{x},t) = -a(\underline{x},t)|\vec{\nabla}z(\underline{x},t)|^2 + c. \tag{7}$$

The constant term c we have added to Eq. (7) [and not present in Eq. (6)] physically represents the uplift of the landscape [9]. c can be simply eliminated by the transformation $z \to z + ct$. The stationary solution of (7) is obtained on setting the right-hand side equal to zero and leads to Eq. (1) [10].

Before discussing the physical $2 + 1$ dimensional case, it is instructive to solve the $1 + 1$ dimensional problem and understand the role of boundary conditions. Ignoring the uplift term, the erosion equation is

$$\dot{z}(x,t) = -x[\partial_x z(x,t)]^2, \qquad x \in [0, L], \tag{8}$$

with boundary conditions

$$\begin{cases} z(x,0) = z_0(x), \\ \dot{z}(L,t) = -c. \end{cases} \tag{9}$$

The outlet is assumed to be at $x = L$, z_0 is monotonically decreasing, and since the basin is between 0 and L, $a(x) = x$. The second equation in (9) balances the uplift. The solution of (8) can be obtained generally for any initial profile $z_0(x)$ $[\partial_x z_0(x) < 0]$ and is given by

$$z(x,t) = -\int_x^L \frac{dx'}{2\sqrt{x'}}\, u(\sqrt{x'},t) - ct, \tag{10}$$

where $u(y,t)$ is given implicitly by

$$u(y,t) = \hat{u}_0\big(y - \tfrac{1}{2}tu(y,t)\big), \tag{11}$$

with

$$\hat{u}_0(y) = \begin{cases} 2y[\partial_x z_0(x)]|_{x=y^2} & y \in [0, \sqrt{L}), \\ -2\sqrt{c} & y \geq \sqrt{L}. \end{cases} \tag{12}$$

The solution $z(x,t,L)$ assumes the scaling form

$$z(x,t,L) = \sqrt{L}\, f\left(\frac{x}{L}, \frac{t}{L^{\zeta}}\right), \tag{13}$$

with $\zeta = \frac{1}{2}$ in $1 + 1$ dimensions. f is a scaling function given by

$$f(w,k) = -\int_w^1 \frac{ds}{2\sqrt{s}}\, \overline{u}(\sqrt{s},k) - ck, \tag{14}$$

where $\overline{u}(\sqrt{x/L}, t/\sqrt{L}) = u(\sqrt{x}, t; L)$.

For example, when $z_0(x) = m(L - x)$ ($m > 0$ represents the slope of the initial landscape), the solution is

$$z(x,t) = \begin{cases} -\frac{mx}{1-mt} + mL, & (x,t) \text{ such that } t \in [0, \overline{t}(x)], \\ 2\sqrt{c}(\sqrt{L} - \sqrt{x}) - ct, & (x,t) \text{ such that } t \geq \overline{t}(x), \end{cases} \tag{15}$$

where the function $\overline{t}(x)$ is determined by imposing continuity of the solution and exists for any $0 < m \leq \sqrt{c/L}$. The solution (15) reaches the stationary state in a finite time $t = t_C = \sqrt{L/c}(2 - \eta) \sim L^{1/2}$ where $\eta = m\sqrt{L/c}$, $0 \leq \eta \leq 1$. For $t \geq t_C$, the scaling function $f(w,k) = 2\sqrt{c}(1 - \sqrt{w}) - ck$.

In two dimensions, the mainstream is topologically one dimensional with the key difference that a is no longer proportional to x but on the average to $x^{1/h}$ where h is the Hack exponent and x is the upstream mainstream length. An effective one dimensional equation for this case becomes

$$\dot{z}(x,t) = -x^{1/h}[\partial_x z(\underline{x},t)]^2, \qquad x \in [0, \mathcal{L}], \tag{16}$$

where $\mathcal{L} \sim L^{d_f}$ is the mainstream length and d_f is the fractal dimension of a stream. The stationary solution is reached after a time $t_C \sim L^{\zeta}$ with $\zeta = d_f(1 - 1/2h)$ and has the scaling form, apart from the drift term

$$z(x) = L^{\zeta} f\left(\frac{x}{L^{d_f}}\right), \tag{17}$$

with $f(z) = \frac{\sqrt{c}}{1-1/2h}(1 - z^{1-1/2h})$ for $h \neq 1/2$ and $f(z) = \sqrt{c}\ln z$ for $h = 1/2$. Our prediction of this scaling form for river basins for which $h = 0.54$–0.60 ought to be amenable to observational verification. We have confirmed that the scaling form holds extremely well in stationary solutions of the two dimensional erosion equation obtained with simulations on a $L \times L$ square lattice (see Figs. 1 and 2) for $L = 32$, 64, and 128.

A direct integration of the two dimensional equation proves to be slow but shows that there are two time scales associated with the dynamics. The first of these is the time taken to determine the connectivity of the spanning tree and is relatively fast, an observation made

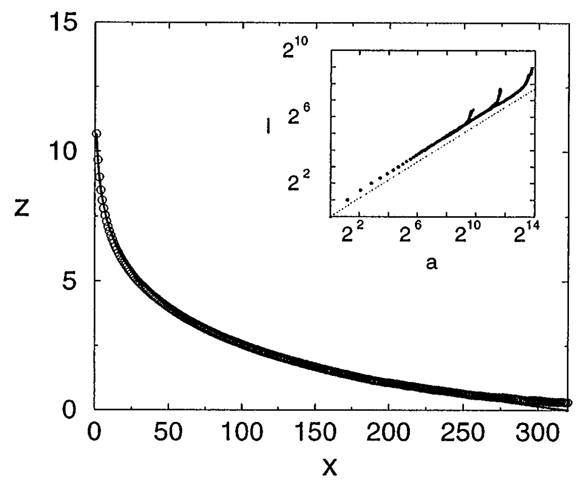

FIG. 1. Profiles along the mainstream (the soil height plotted against the length measured from the source along the main stream) obtained in stationary solutions of the two dimensional erosion equation on a 128×128 square lattice and averaged over 100 samples starting from different randomly chosen initial conditions are plotted together with the analytical result [Eq. (17)] with $h = 0.55$, $d_f = 1.1$, and $c = 1$, as in the simulation. The value of h (0.55 ± 0.02) was deduced from the slope of the log-log plot of the upstream lengths along the mainstream versus the corresponding areas shown in the inset. d_f (1.1 ± 0.04) was determined by a collapse (not shown) of the distributions of upstream lengths for various system sizes.

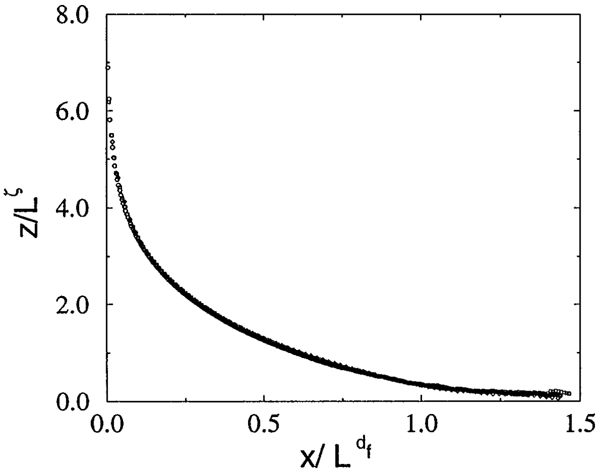

FIG. 2. Collapse of profiles along the mainstream corresponding to 32×32, 64×64, and 128×128 square lattices obtained with $\zeta = d_f(1 - 1/2h)$. The exponent values are the same as in the caption for Fig. 1.

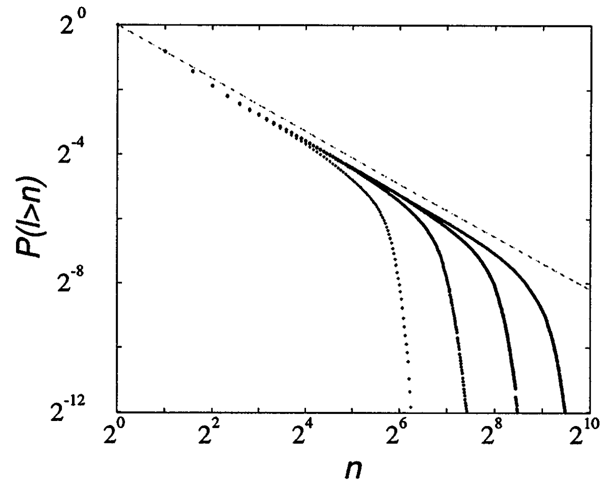

FIG. 4. Distribution of upstream lengths with the same lattice sizes and statistics as in Fig. 3. The slope $\gamma - 1 = 0.82 \pm 0.02$ is in excellent accord with the scaling prediction (3).

earlier by Sinclair and Ball [7]. The second involves further erosion (without changing the spanning tree) until the soil height acquires a stable profile that satisfies Eq. (1). This may account for the robustness of the scaling statistics associated with the spanning tree, as the imprinting of the tree occurs relatively early in the evolution process. Most of the computational runs in two dimensions were carried out not by dynamical integration but by an iterative procedure. Specifically one begins with an arbitrary spanning tree, determines a_i and uses Eq. (1) to construct a landscape. Steepest descent is then employed to obtain a new spanning tree. This procedure is iterated to self-consistency. The exponents and their scaling relationship are found to be in excellent accord with observational data (Figs. 3 and 4). A picture of a typical landscape is shown in Fig. 5.

The optimal channel network (OCN) [11] is a selection procedure which postulates that nature selects the spanning tree that minimizes the total energy dissipated $E_D = \sum_i \sqrt{a_i}$. Studies of the statistics of optimal trees by seeking a local minimum of E_D (a local minimum corresponds to an "optimal" tree such that any attempt at flipping one of the local outlet bonds to a new configuration while preserving the spanning tree geometry would lead to an increase in E_D) have yielded consistent scaling relationships and excellent quantitative accord with the measured exponents τ, γ, and h [1,12]. It can be proved (the details will be presented elsewhere) that the OCN is a stationary solution of the erosion equation. The basic idea is to start with an optimal tree, determine the a_i, and use Eq. (1) to construct the landscape. The condition for the

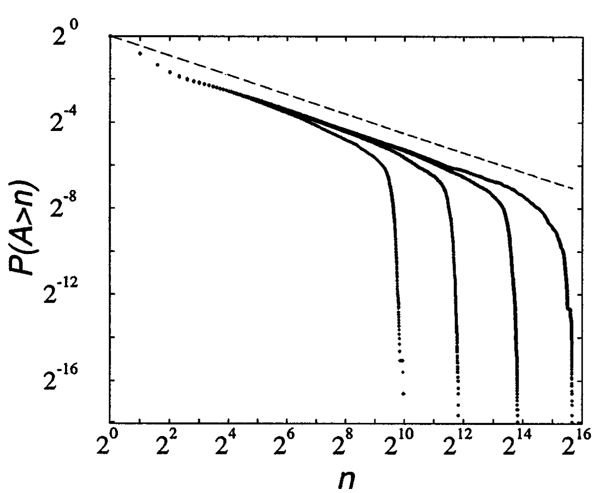

FIG. 3. Distribution of accumulated areas averaged over 100 samples on 32×32, 64×64, 128×128, and 256×256 square lattices. The slope $\tau - 1 = 0.45 \pm 0.02$ is in excellent accord with the scaling prediction of $1 - h$.

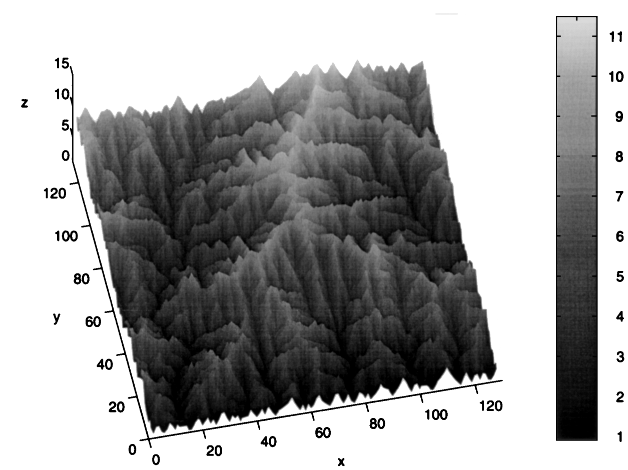

FIG. 5. A typical landscape obtained by the iterative procedure described in the text for a 128×128 square lattice. The different shades correspond to different heights according to the bar on the right-hand side of the figure. Periodic boundary conditions have been chosen in the x direction. Outlets lie on the $y = 0$ line. $z(x, y)$ represents the soil height.

local minima of E_D can then be used to show that if one uses steepest descent, one recovers the original tree, and self-consistency is attained.

An analytical study [5] of the statistics associated with the spanning tree corresponding to the global minimum of E_D has lead to values of the exponents that are inconsistent with observations. This reinforces the idea of feasible optimality [13] which in the context of the new results obtained herein states that when the landscape of the dissipated energy is riddled with local minima, the global minimum (or minima) forms a set of negligible measure (possibly zero) and the stationary solutions of the erosion equation would correspond to local minima. What is remarkable is that these accessible stationary states show consistent scaling statistics over a few orders of magnitude and exponents that are in perfect agreement with observations.

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Note added.—After this work was submitted for publication, we became aware of a closely related preprint by Somfai and Sander [14] in which a Landau theory for erosion was presented that leads to a universal form for the large scale behavior of river networks.

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[9] The existence of an uplift originated by tectonic forces is well known in geomorphology. A landscape represents the instantaneous equilibrium of two concurrently active processes: uplift (endogenic) and degradation (exogenic). A stationary state results from the exact balance of these two agents.

[10] Equation (1) holds on average, while Eq. (7) holds in a deterministic sense. We have explicitly verified that the scaling behavior and the conclusions of this paper are unchanged on allowing for random rainfall.

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