Pseudospin as a Relativistic Symmetry

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(Received 8 October 1996)

We show that pseudospin symmetry in nuclei could arise from nucleons moving in a relativistic mean field which has an attractive scalar and repulsive vector potential nearly equal in magnitude. [S0031-9007(96)02176-X]

PACS numbers: 21.60.Cs, 21.10.–k, 21.30.Fe, 24.10.Jv

Almost 30 years ago a quasidegeneracy was observed in heavy nuclei between single-nucleon doublets with quantum numbers $(n_r, \ell, j = \ell + \frac{1}{2})$ and $(n_r - 1, \ell + 2, j = \ell + \frac{3}{2})$ where n_r, ℓ, and j are the single nucleon radial, orbital, and total angular momentum quantum numbers, respectively [1,2]. These authors defined a "pseudo" orbital angular momentum $\tilde{\ell} = \ell + 1$; for example, $(n_r s_{1/2}(n_r - 1) d_{3/2})$ will have $\tilde{\ell} = 1$, $(n_r p_{3/2}, (n_r - 1) f_{5/2})$ will have $\tilde{\ell} = 2$, etc. Then these doublets are almost degenerate with respect to "pseudo" spin, $\tilde{s} = \frac{1}{2}$, since $j = \tilde{\ell} \pm \tilde{s}$ for the two states in the doublet. This symmetry has been used to explain a number of phenomena in nuclear structure [3] including most recently the identical rotational bands observed in nuclei [4]. Despite this long history of pseudospin symmetry [5,6], the origin of this symmetry has eluded explanation. Recently it was shown [7] that relativistic mean field theories predict the correct spin-orbit splitting [8]. In this paper we identify a possible reason for this; namely that the symmetry arises from the near equality in magnitude of an attractive scalar, $-V_s$, and repulsive vector, V_v, relativistic mean fields, $V_s \sim V_v$, in which the nucleons move. Such a near equality of mean fields follows from relativistic field theories with interacting nucleons and mesons [8], with nucleons interacting via Skyrme-type interactions [9], and from QCD sum rules [10].

A nucleon moving in a spherical field has the total angular momentum j, its projection on the z axis, m, and $\hat{\kappa} = -\hat{\beta}(\hat{\sigma} \cdot \hat{L} + 1)$ conserved, where $\hat{\beta}$ is the Dirac matrix [11]. The eigenvalues of $\hat{\kappa}$ are $\kappa = \pm(j + \frac{1}{2})$; $-$ for aligned spin ($s_{1/2}, p_{3/2}$, etc.) and $+$ for unaligned spin ($p_{1/2}, d_{3/2}$, etc.). Hence we use the quantum number κ since it is sufficient to label the orbitals. The Dirac equation for the single-nucleon radial wave function (g_κ, f_κ) in dimensionless units is given by [11]

$$\left[\frac{d}{dr} + \frac{1+\kappa}{r}\right] g_\kappa = [2 - E - V(r)] f_\kappa \,, \quad (1)$$

$$\left[\frac{d}{dr} + \frac{1-\kappa}{r}\right] f_\kappa = [E - \Delta(r)] g_\kappa \,, \quad (2)$$

where r is the radial coordinate in units of length $\hbar c/mc^2$, $V(r) = [V_v(r) + V_s(r)]/mc^2$, $\Delta(r) = [V_s(r) - V_v(r)]/mc^2$, and E is the binding energy ($E > 0$) of the nucleon in units of the nucleon mass mc^2. First we show that, in the limit of equality of the magnitude of the vector and scalar potential $\Delta(r) = 0$, pseudospin is exactly conserved. To do this we solve for g_κ in (2) and substitute into (1), obtaining the second order differential equation for f_κ,

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{\tilde{\ell}(\tilde{\ell}+1)}{x^2} + [V(r) - 2 + E]\right] \times f_\kappa = 0 \,, \quad (3)$$

where $x = \sqrt{E}\, r$ and

$$\tilde{\ell} = \kappa - 1, \ \kappa > 0; \quad \tilde{\ell} = -\kappa, \ \kappa < 0 \,, \quad (4)$$

which agrees with the original definition of the pseudo-orbital angular momentum [1,2]. For example, for $[n_r s_{1/2}, (n_r - 1) d_{3/2}]$, $\kappa = -1$ and 2, respectively, giving $\tilde{\ell} = 1$ in both cases. Futhermore, the physical significance of $\tilde{\ell}$ is revealed; it is the "orbital angular momentum" of the lower component of the Dirac wave function.

Eq. (3) is a Schrödinger equation with an attractive potential V and binding energy $2 - E$ which depends only on the pseudo-orbital angular momentum $\tilde{\ell}$ through the pseudo rotational kinetic energy $\tilde{\ell}(\tilde{\ell} + 1)/x^2$, and not on κ. Hence the eigenenergies and eigenfunction component f_κ do not depend on κ but only on $\tilde{\ell}$. Thus the doublets with the same $\tilde{\ell}$ but different κ ($\kappa = \tilde{\ell} + 1$ and $\kappa = -\tilde{\ell}$) will be degenerate, producing pseudospin symmetry.

However, in this limit there will not be any bound Dirac valence states, only Dirac sea states, which contradicts reality. Is it possible that we can have bound valence states *and* quasidegeneracy for a small $\Delta(r)$? To answer that question we look at two examples. First is the spherical Coulomb potential and the second is the spherical potential well.

The spherical Coulomb potential for arbitrary scalar and vector fields, $V_{s,v}(r) = \alpha_{s,v}/r$, can be solved analytically [11]. The valence eigenenergies are given by

$$E_{n,\kappa} = \frac{4(n+\lambda)^2[1 - \sqrt{(1 - (\alpha\delta)/(n+\lambda)^2}] - 2\delta(\alpha - \delta)}{(\alpha - \delta)^2 + 4(n+\lambda)^2} \,, \quad (5)$$

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where n is the principal quantum number, $n = 1, 2, \ldots, \alpha = \alpha_s + \alpha_v$, $\delta = \alpha_s - \alpha_v$, and $\lambda = |\kappa| \times (\sqrt{1 + \alpha\delta/\kappa^2} - 1)$. The allowed values of κ are $\kappa = \pm 1, \pm 2, \ldots, \pm(n-1), -n$. The dependence on κ is in λ. If the scalar and vector potential are equal, $\delta = 0$, then the binding energy vanishes, $E_{n,\kappa} = 0$, and hence no bound valence states as stated earlier. For δ small,

$$E_{n,\kappa} \approx \frac{\delta^2}{2n^2}\left(1 + \frac{\alpha\delta(2n^2(|\kappa| - 2) - \alpha^2)}{|\kappa| n^2(4n^2 + \alpha^2)} + \ldots\right). \tag{6}$$

Thus the pseudospin symmetry is broken in third order in δ. We notice that the breaking decreases as n increases and, for a given n, the state with the largest $|\kappa|$ (which means the pseudospin partner with $\kappa > 0$) will have the largest binding energy. Thus pseudospin quasidegeneracy coexists with an infinite number of bound valence Dirac states.

However, the Coulomb potential is not realistic for nuclei and, furthermore, the Coulomb potential has higher degeneracies than pseudospin since the energies depend only on n, and not κ in the lowest order. For these reasons we turn now to the spherical potential well:

$$V_{s,v}(r) = V_{s,v} > 0,\ r < R;\ V_{s,v}(r) = 0,\ r > R\,. \tag{7}$$

The solution of the Dirac equation (1), (2) is given in terms of spherical Bessel functions for $r < R$,

$$g_\kappa = A\mathbf{j}_{\tilde{\ell}+\tilde{\epsilon}}(z),\ f_\kappa = \frac{\tilde{\epsilon}Ak}{2 - E - V}\mathbf{j}_{\tilde{\ell}}(z),\ r < R \tag{8}$$

and modified spherical Bessel functions for $r > R$ [11],

$$g_\kappa = \overline{A}\,\mathbf{k}_{\tilde{\ell}+\tilde{\epsilon}}(y),\ f_\kappa = -\frac{\overline{A}K}{2 - E}\mathbf{k}_{\tilde{\ell}}(y),\ r > R\,, \tag{9}$$

where $z = kr$, $y = Kr$, and the wave numbers are given by

$$k^2 = (\Delta - E)(2 - E - V) > 0,\ K^2 = E(2 - E) > 0\,, \tag{10}$$

where $\tilde{\epsilon} = \kappa/|\kappa|$ is the pseudohelicity, since $j = \tilde{\ell} + \tilde{\epsilon}\frac{1}{2}$, and the eigenvalues of $\tilde{\epsilon}$ are ± 1.

The two solutions must match at the boundary $r = R$, leading to the two conditions which determine the eigenvalues for the same $\tilde{\ell}$, but different κ:

$$\frac{Z_\kappa \mathbf{j}_{\tilde{\ell}+1}(Z_\kappa)}{\mathbf{j}_{\tilde{\ell}}(Z_\kappa)} = -\frac{Y_\kappa(\Delta - E_\kappa)\mathbf{k}_{\tilde{\ell}+1}(Y_\kappa)}{E_\kappa \mathbf{k}_{\tilde{\ell}}(Y_\kappa)},\ \kappa > 0\,, \tag{11}$$

$$\frac{Z_\kappa \mathbf{j}_{\tilde{\ell}-1}(Z_\kappa)}{\mathbf{j}_{\tilde{\ell}}(Z_\kappa)} = \frac{Y_\kappa(\Delta - E_\kappa)\mathbf{k}_{\tilde{\ell}-1}(Y_\kappa)}{E_\kappa \mathbf{k}_{\tilde{\ell}}(Y_\kappa)},\ \kappa < 0\,, \tag{12}$$

where $Z = kR$ and $Y = KR$. Since there are two different equations for the states with the same $\tilde{\ell}$ but different $\tilde{\kappa}$, the eigenenergies of these two different states will be different in general.

Using the recurrence relations between Bessel functions

$$Z\mathbf{j}_{\tilde{\ell}+1}(Z) = (2\tilde{\ell} + 1)\mathbf{j}_{\tilde{\ell}}(Z) - Z\mathbf{j}_{\tilde{\ell}-1}(Z);$$
$$Y\mathbf{k}_{\tilde{\ell}+1}(Y) = (2\tilde{\ell} + 1)\mathbf{k}_{\tilde{\ell}}(Y) + Y\mathbf{k}_{\tilde{\ell}-1}(Y)\,, \tag{13}$$

we can eliminate $\mathbf{j}_{\tilde{\ell}-1}, \mathbf{k}_{\tilde{\ell}-1}$ and rewrite these equations as

$$-\frac{Z_\kappa \mathbf{j}_{\tilde{\ell}+1}(Z_\kappa)}{\mathbf{j}_{\tilde{\ell}}(Z_\kappa)} = \frac{Y_\kappa(\Delta - E_\kappa)\mathbf{k}_{\tilde{\ell}+1}(Y_\kappa)}{E_\kappa \mathbf{k}_{\tilde{\ell}}(Y_\kappa)}\,,\ \kappa > 0\,, \tag{14}$$

$$-\frac{Z_\kappa \mathbf{j}_{\tilde{\ell}+1}(Z_\kappa)}{\mathbf{j}_{\tilde{\ell}}(Z_\kappa)} = \frac{Y_\kappa(\Delta - E_\kappa)\mathbf{k}_{\tilde{\ell}+1}(Y_\kappa)}{E_\kappa \mathbf{k}_{\tilde{\ell}}(Y_\kappa)} - (2\tilde{\ell} + 1)\frac{\Delta}{E_\kappa}\,,\ \kappa < 0\,, \tag{15}$$

thereby displaying the fact that the equations become identical for $\Delta = 0$ producing the pseudospin degeneracy but, as we shall see, no Dirac valence bound states.

In Fig. 1 we plot the left-hand side (LHS) of (14), (15) as a function of Z. The LHS decreases from a value of zero at $Z = 0$ to negative infinity at $Z^{(0)}_{1,\tilde{\ell}}$, where $Z^{(0)}_{n,\tilde{\ell}}$ is the nth zero of the spherical Bessel function $j_{\tilde{\ell}}(Z^{(0)}_{n,\tilde{\ell}}) = 0$, with $Z = 0$ corresponding to $n = 0$. The LHS then becomes discontinuous at this point, and for $Z > Z^{(0)}_{1,\tilde{\ell}}$ it decreases from positive infinity to zero at $Z^{(0)}_{1,\tilde{\ell}+1}$, and then negative infinity at $Z^{(0)}_{2,\tilde{\ell}}$ and so on. We call the region with $Z^{(0)}_{n,\tilde{\ell}} \leq Z < Z^{(0)}_{n+1,\tilde{\ell}}$ the nth branch.

On the other hand, the right-hand side (RHS) of both (14) and (15) increases monotonically, as illustrated in Fig. 2. The eigenvalues are determined by the points of intersections in the nth branch $Z_{n,\tilde{\kappa}}$, giving the valence eigenenergies

$$E_{n_r,\kappa} = \frac{2 - V + \Delta}{2} - \sqrt{\left(\frac{2 - V - \Delta}{2}\right)^2 + \left(\frac{Z_{n,\kappa}}{R}\right)^2}\,,$$
$$n_r = n - 1,\ \kappa > 0,\ n_r = n,\ \kappa < 0\,, \tag{16}$$

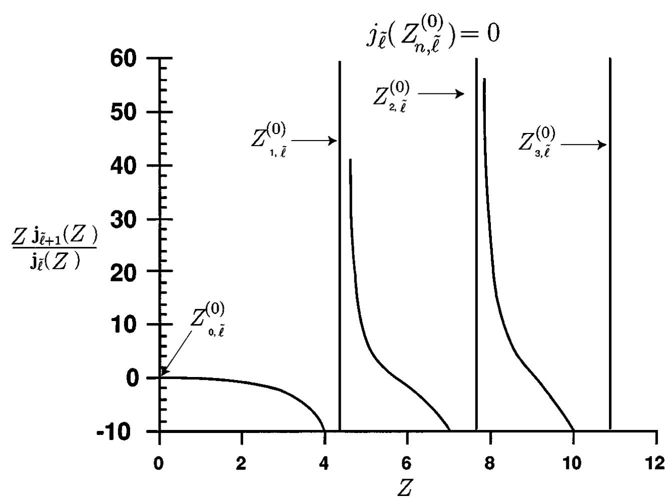

FIG. 1. The LHS of (14), (15) plotted versus Z; $Z^{(0)}_{n,\tilde{\ell}}$ is the nth zero of $j_{\tilde{\ell}}(Z^{(0)}_{n,\tilde{\ell}}) = 0$.

where the radial quantum number will become clear subsequently. The RHS of (14) increases from zero at $Z = 0$ ($E = \Delta$) to infinity at $Z = Z_{\max} = \sqrt{\Delta(2 - V)}R(E = 0)$ [see (16)]. However, the RHS of (15) is smaller by an amount $(2\tilde{\ell} + 1)\Delta/E$, and increases from $-(2\ell + 1)$ at $Z = 0$ ($E = \Delta$) to a constant at $Z = Z_{\max}$ ($E = 0$). This means that for $\kappa < 0$, there will be a bound state for each branch in Fig. 2 as long as $Z_{n,\kappa} < Z_{\max}$. Furthermore, the upper component for $\kappa < 0, \mathbf{j}_{\tilde{\ell}-1}$, has a zero in each of these branches and thus the radial quantum number is then $n_r = n = 0, 1, \ldots$. However, for $\kappa > 0$ there will not be a bound state for $n = 0$ since $\Delta/E = 1$ at $Z = 0$, and thus the two curves intersect only at $Z = 0$, which means $k = 0$, and hence there is no bound state for $n = 0$ [see (8)]. However, there will be a bound state for all the other branches in Fig. 2 as long as $Z_{n,\kappa} < Z_{\max}$. Furthermore, the upper component for $\kappa > 0, \mathbf{j}_{\tilde{\ell}+1}$ does not have a zero in the $n = 0$ branch but does have a zero in the other branches so the radial quantum number is then $n_r = n - 1 = 0, 1, \ldots$. This means that the orbit with $n_r = 0, \kappa < 0$ does not have pseudospin partner (in fact this orbital is the "intruder" orbital observed in heavy nuclei), but the orbits with $n_r, \kappa < 0, n_r - 1, \kappa > 0$ are in the same branch and are thus pseudospin partners, which agrees with experiment. Also we see from Fig. 2 That the RHS for $\kappa > 0$ intersects the LHS at a smaller Z than $\kappa < 0$, and thus $E_{n_r-1,\kappa>0} > E_{n_r,\kappa<0}$ for the same $\tilde{\ell}$ in agreement with experiment. Furthermore, as the RHS for both κ increases, they intersect the LHS at points in which the LHS has a larger slope, and therefore the points of intersection $Z_{n,\kappa}$ are closer. Thus these pseudospin partners become closer in energy as the radial quantum number increases.

These features can be seen in the limit of a large scalar potential $V_s \lesssim 1$. In this limit the RHS of (14), (15) is large. We use the Bessel function identity [12]

$$-\frac{\mathbf{j}_{\tilde{\ell}+1}(Z)}{\mathbf{j}_{\tilde{\ell}}(Z)} = \sum_{p=1}^{\infty}\left[\frac{1}{Z - Z^{(0)}_{p,\tilde{\ell}}} + \frac{1}{Z + Z^{(0)}_{p,\tilde{\ell}}}\right], \qquad (17)$$

which, in the nth branch and for the LHS large and positive, can be approximated as $-\mathbf{j}_{\tilde{\ell}+1}(Z)/\mathbf{j}_{\tilde{\ell}}(Z) \approx 1/(Z - Z^{(0)}_{n,\tilde{\ell}})$. If, in addition, $Y^{(0)}_{n,\tilde{\ell}}$ is large,

$$E_{n_r-1,\kappa>0} \approx E^{(0)}_{n,\tilde{\ell}}\,, \qquad (18)$$

where we have denoted $E^{(0)}_{n,\tilde{\ell}}, Y^{(0)}_{n,\tilde{\ell}}$ as the values of E, Y for $Z = Z^{(0)}_{n,\tilde{\ell}}$, and

$$E_{n_r-1,\kappa>0} - E_{n_r,\kappa<0} \approx \frac{(2\tilde{\ell} + 1)\Delta E^{(0)}_{n,\tilde{\ell}}}{(Y^{(0)}_{n,\tilde{\ell}})^2(\Delta - E^{(0)}_{n,\tilde{\ell}})}\,. \qquad (19)$$

Thus we see that the energy splitting decreases as the binding energy decreases, which is consistent with the fact that pure pseudospin symmetry occurs when there are no bound Dirac valence states, and that the splitting decreases as the radial quantum number increases. Furthermore, for states within the same major shell, the splitting decreases as the pseudo-orbital angular momentum decreases.

Hence we have shown that pseudospin quasidegeneracy in heavy nuclei can be explained by the fact that nucleons in a nucleus move in an attractive scalar, $-V_s$, and repulsive vector, V_v, relativistic mean fields, which are nearly equal in magnitude, $V_s \sim V_v$. The energy splitting between states with the same pseudo-orbital angular momentum $\tilde{\ell}$ decreases as the binding energy decreases and as $\tilde{\ell}$ decreases. Although such a near equality of mean fields has been derived in specific relativistic field theories [8,9], this result probably is a general feature of any relativistic model which fits nuclear binding energies, and hence very likely a general feature independent of any one model [10]. In [9] it was shown that $V_s \sim V_v$ for the isoscalar part of the nuclear mean field (the largest part) but not for the isovector part, and the isovector potential has a different shape than the isoscalar potential. This implies that pseudospin symmetry may be enhanced in heavy proton-rich nuclei, with $N \sim Z$; these nuclei shall be measured in new radioactive beam facilities.

Pseudospin symmetry has been observed also in deformed nuclei [4]; we are investigating the deformed Dirac equation as well. Also the explanation espoused in this paper implies a connection between the wave functions of the pseudospin doublets. This relationship is being worked out.

The author thanks B. Serot for discussions and N. Walet for making the author aware of Ref. [10].

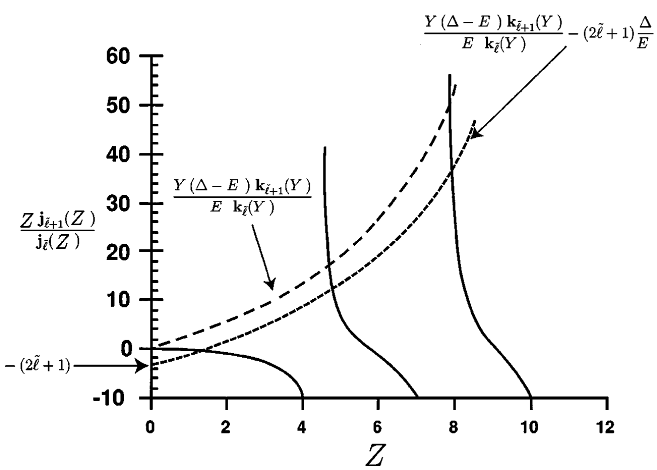

FIG. 2. The LHS of (14), (15) (solid line), the RHS of (14) (dashed line), and the RHS of (15) (short dashed line) plotted versus Z for $\tilde{\ell} = 1, V = 1.7, \Delta = 0.3, R = 33.5$, the radius of ^{208}Pb in dimensionless units.

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