From Phase to Lag Synchronization in Coupled Chaotic Oscillators

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(Received 15 October 1996)

We study synchronization transitions in a system of two coupled self-sustained chaotic oscillators. We demonstrate that with the increase of coupling strength the system first undergoes the transition to phase synchronization. With a further increase of coupling, a new synchronous regime is observed, where the states of two oscillators are nearly identical, but one system lags in time to the other. We describe this regime as a state with correlated amplitudes and a constant phase shift. These transitions are traced in the Lyapunov spectrum. [S0031-9007(97)03271-7]

PACS numbers: 05.45.+b

Synchronization phenomena in coupled chaotic systems have been extensively studied in the context of laser dynamics [1], electronic circuits [2,3], chemical and biological systems [4], and secure communication [5]. Complete, generalized, and phase synchronizations of chaotic oscillators have been described theoretically and observed experimentally. Complete (full) synchronization (CS) implies coincidence of states of interacting systems, $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ [6–8]; it appears only if interacting systems are identical. Otherwise, if the parameters of coupled oscillators slightly mismatch, the states are close $|\mathbf{x}_1(t) - \mathbf{x}_2(t)| \approx 0$ but remain different [7,9]. A generalized synchronization (GS) [10], introduced for drive-response systems, is defined as the presence of some functional relation between the states of response and drive, i.e., $\mathbf{x}_2(t) = \mathcal{F}[\mathbf{x}_1(t)]$ [11]. The phase synchronization (PS) described in [12,13] and experimentally observed in [14] means entrainment of phases of chaotic oscillators, whereas their amplitudes remain chaotic and noncorrelated; the notion of phase is discussed in details in [15]. The relation between these different types of synchronization and the scenarios of transitions to or between them have not been addressed yet.

In this Letter we study synchronization of symmetrically coupled *nonidentical* oscillators. We demonstrate that, with the increase of coupling, first the transition from nonsynchronous state to PS occurs. For larger couplings a new regime which we call lag synchronization (LS) is observed. LS appears as a coincidence of *shifted in time* states of two systems, $\mathbf{x}_1(t + \tau_0) = \mathbf{x}_2(t)$. Finally, with a further increase of coupling, the time shift decreases and this regime tends to CS. We show that these transitions are related to the changes in the spectrum of Lyapunov exponents (LE).

Synchronization is a universal nonlinear phenomenon, and its main features are typically independent of particular properties of a model. As a first example, we study two coupled Rössler systems [16],

$$\begin{aligned}\dot{x}_{1,2} &= -\omega_{1,2} y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}),\\ \dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + a y_{1,2},\\ \dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c),\end{aligned} \tag{1}$$

where $a = 0.165$, $f = 0.2$, and $c = 10$. The parameters $\omega_{1,2} = \omega_0 \pm \Delta$ and ε determine the mismatch of natural frequencies and the coupling, respectively. These equations serve as a good model for real systems having a strange attractor that appears via period-doubling cascade, e.g., for electronic circuits [2,3] or chemical systems [17].

To describe the phase and the lag synchronization, we need to introduce corresponding quantities. For the Rössler attractor the phase and the amplitude can be conveniently introduced as [13,15,17]

$$\phi = \arctan \frac{y}{x}, \quad A = (x^2 + y^2)^{1/2}. \tag{2}$$

The phase can be easily calculated for each subsystem, thus allowing one to determine mean frequencies $\Omega_{1,2} = \langle \dot{\phi}_{1,2} \rangle$ and relations of locking between them. To characterize LS, we introduce a similarity function S as a time averaged difference between the variables x_1 and x_2 (with mean values being subtracted) taken with the time shift τ [18],

$$S^2(\tau) = \frac{\langle [x_2(t + \tau) - x_1(t)]^2 \rangle}{[\langle x_1^2(t) \rangle \langle x_2^2(t) \rangle]^{1/2}}, \tag{3}$$

and search for its minimum $\sigma = \min_\tau S(\tau)$. If the signals x_1 and x_2 are independent, the difference between them is of the same order as the signals themselves; respectively, $S(\tau) \sim 1$ for all τ. If $x_1(t) = x_2(t)$, as in the case of CS, $S(\tau)$ reaches its minimum $\sigma = 0$ for $\tau = 0$. Below, we demonstrate a nontrivial case, when the similarity function $S(\tau)$ has a minimum for nonzero time shift τ, meaning a time lag exists between the two processes.

First, we describe the transition to PS in the system (1) (see also [12]). The parameters $\omega_0 = 0.97$ and $\Delta = 0.02$ are chosen by trial in such a way that appearance of large windows of periodic behavior is avoided. The calculation of the average frequencies $\Omega_{1,2}$ allows us to follow the transition at $\varepsilon = \varepsilon_p \approx 0.036$ to the frequency entrainment $\Omega_1 = \Omega_2 = \Omega$ (see Fig. 1). Because of high coherence of the Rössler attractor, the phase difference in the synchronous regime is bounded and oscillates around some mean value $\delta\phi = \langle \phi_1(t) - \phi_2(t) \rangle \neq 0$.

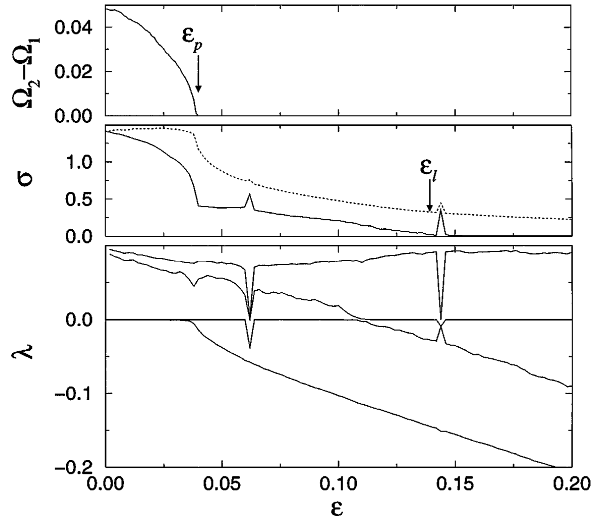

FIG. 1. The frequency difference $\Omega_1 - \Omega_2$, the minimum of the similarity function σ, and the four largest Lyapunov exponents λ of two coupled Rössler oscillators vs the coupling ε. Three different regions are clearly seen on the σ vs ε plot correspondent to a nonsynchronous state, phase, and lag synchronization, respectively. The transitions between these regimes are reflected in the spectrum of Lyapunov exponents: At the first transition, one of the zero LE becomes negative, while the second transition corresponds to the zero crossing of one of the positive LE. The dashed line shows the dependence of $S(0)$ on the coupling; from this plot one can see that comparison of states of interacting systems without time shift does not reveal the transition to LS. Two "outbursts" on the σ vs ε plot at $\varepsilon \approx 0.06$ and $\varepsilon \approx 0.145$ correspond to period 3 windows.

For stronger coupling $\varepsilon = \varepsilon_l \approx 0.14$ we observe a new transition to lag synchronization (see the σ vs ε curve in Fig. 1). In Fig. 2 we show numerically obtained similarity functions in system (1) for relatively weak, intermediate, and strong coupling. For weak coupling $\varepsilon < \varepsilon_p$ (curves 1 and 2), $S \sim 1$ and practically does not depend on τ, as can be expected for independent signals. For intermediate coupling strength $\varepsilon_p < \varepsilon < \varepsilon_l$, a minimum of $S(\tau)$ appears (curves 3 and 4) indicating the existence of some characteristic time shift τ_0 between x_1 and x_2. This shift is related to the phase difference as $\tau_0 = \delta\phi/\Omega$. Note that in this regime the amplitudes are uncorrelated, so the value of $S(\tau_0)$ is relatively large. Further increase of coupling makes, at $\varepsilon \approx \varepsilon_l$, this minimum very sharp (curves 5 and 6) and practically equal to zero. It means that the states of the systems become identical, but shifted in time with respect to each other. The regime of LS is clearly demonstrated in Fig. 3 by plotting $x_1(t + \tau_0)$ vs $x_2(t)$. It is important that calculations of $S(0)$, i.e., the comparison of x_1 and x_2 without time shift, reveal no transition at $\varepsilon = \varepsilon_l$. For larger couplings $\varepsilon > \varepsilon_l$, the time lag τ_0 continuously decreases, but no qualitative transitions are observed.

The transitions between different types of synchronization can be related to the changes in the Lyapunov spectrum (see Fig. 1). For small coupling $\varepsilon < \varepsilon_p$, there are two positive LE (corresponding to chaotic amplitudes) and two nearly zero LE (corresponding to independently rotating phases). At the phase locking transition at $\varepsilon \approx \varepsilon_p$, one of the zero LEs becomes negative, corresponding to a definite stable relation between phases (one zero LE, corresponding to a simultaneous phase shift of both Rössler oscillators, remains for all couplings, as it should in an autonomous system) [12]. The second transition to

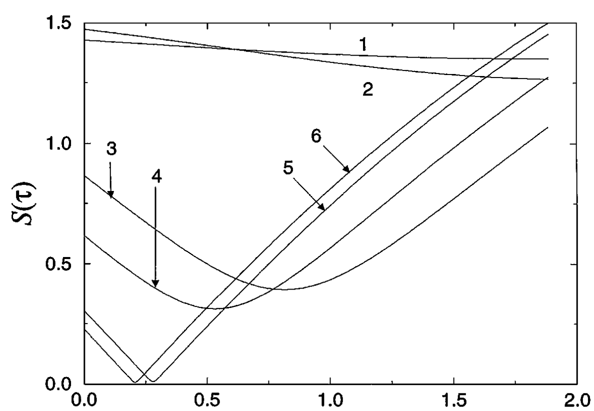

FIG. 2. Similarity function $S(\tau)$ for different values of coupling strength ε (1: $\varepsilon = 0.01$, 2: $\varepsilon = 0.015$, 3: $\varepsilon = 0.05$, 4: $\varepsilon = 0.075$, 5: $\varepsilon = 0.15$, 6: $\varepsilon = 0.2$). With the increase of coupling, a minimum appears, indicating the existence of a certain phase shift between interacting systems (curves 3 and 4). In the regime of lag synchronization (curves 5 and 6), the minimum is extremely small.

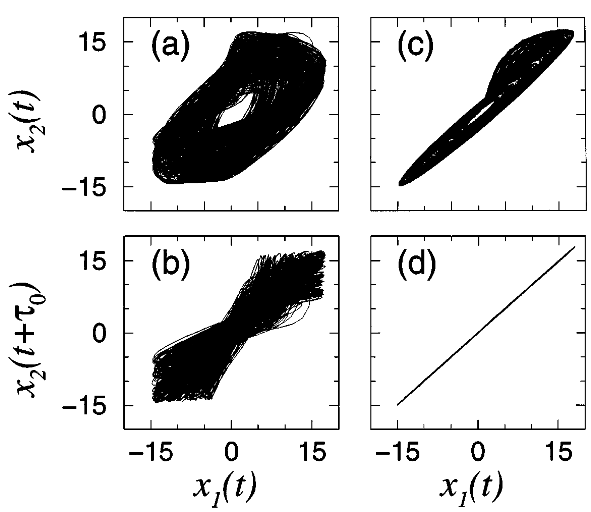

FIG. 3. Projections of the attractor of the coupled system on the plane $(x_1(t), x_2(t))$ and delayed-coordinate plots $x_2(t + \tau_0)$ vs $x_1(t)$ for different values of coupling. (a),(b) $\varepsilon = 0.05$, a regime with phase synchronization, (c),(d) $\varepsilon = 0.2$, a regime with lag synchronization. The qualitative difference between PS and LS is clearly seen from (b),(d), where time shifts, $\tau_0 = 0.87$ and $\tau_0 = 0.21$, respectively, correspond to the minima of the similarity function $S(\tau)$. The panel (d) demonstrates that the state of one of the oscillators is delayed in time with respect to the other; the same can be shown for the variables $y_{1,2}$ and $z_{1,2}$ as well.

LS corresponds to the change of the sign by the second positive LE, but does not exactly coincide with it due to the intermittency discussed below. This means that the relation appears not only between the phases but also between the amplitudes. The phase shift remains, and therefore a time lag between the signals x_1 and x_2 is observed.

To develop an approximate theory of the phase and lag synchronization in the model (1), let us rewrite it in the variables (2):

$$\begin{aligned}\dot{A}_{1,2} &= aA_{1,2}\sin^2\phi_{1,2} - z_{1,2}\cos\phi_{1,2} \\ &\quad + \varepsilon(A_{2,1}\cos\phi_{2,1}\cos\phi_{1,2} - A_{1,2}\cos^2\phi_{1,2}), \\ \dot{\phi}_{1,2} &= \omega_{1,2} + a\sin\phi_{1,2}\cos\phi_{1,2} + z_{1,2}/A_{1,2}\sin\phi_{1,2} \\ &\quad - \varepsilon(A_{2,1}/A_{1,2}\cos\phi_{2,1}\sin\phi_{1,2} - \cos\phi_{1,2}\sin\phi_{1,2}), \\ \dot{z}_{1,2} &= f - cz_{1,2} + A_{1,2}z_{1,2}\cos\phi_{1,2}.\end{aligned} \tag{4}$$

The main idea in analyzing this system is to use averaging over rotations of the phases $\phi_{1,2}$, assuming that the amplitudes vary slowly. Although there is no small parameter allowing one to perform this procedure mathematically correct, we will see that the results correspond rather well to the properties of the full system. Introducing the "slow" phases $\theta_{1,2}$ according to $\phi_{1,2} = \omega_0 t + \theta_{1,2}$, and averaging the equations for them, we get

$$\frac{d}{dt}(\theta_1 - \theta_2) = 2\Delta - \frac{\varepsilon}{2}\left(\frac{A_2}{A_1} + \frac{A_1}{A_2}\right)\sin(\theta_1 - \theta_2). \tag{5}$$

When we neglect the fluctuations of the amplitudes on the right-hand side, this equation has a stable fixed point

$$\theta_1 - \theta_2 = \arcsin\frac{4\Delta A_1 A_2}{\varepsilon(A_2^2 + A_1^2)} \tag{6}$$

which corresponds to the phase locking of the Rössler systems. The transition point to phase synchronization can thus be estimated as $\varepsilon_p \approx 4\Delta\langle A_1A_2/(A_2^2 + A_1^2)\rangle$. If we neglect the variations of the amplitudes we obtain $\varepsilon_p \approx 2\Delta = 0.04$ (for the parameters used), in rough agreement with the numerical result $\varepsilon_p \approx 0.036$.

Now we turn to the description of the next transition, and for this purpose we assume constant slow phases in the equations for A and z. Here we also perform the averaging, except for the terms containing both the fast phases $\phi_{1,2}$ and the variables $z_{1,2}$, because the latter, contrary to the amplitudes, cannot be considered as slow. As a result we obtain

$$\begin{aligned}\dot{A}_{1,2} &= \frac{a}{2}A_{1,2} - z_{1,2}\cos(\omega_0 t + \theta_{1,2}) \\ &\quad + \frac{\varepsilon}{2}[A_{2,1}\cos(\theta_1 - \theta_2) - A_{1,2}], \\ \dot{z}_{1,2} &= f - cz_{1,2} + A_{1,2}z_{1,2}\cos(\omega_0 t + \theta_{1,2}).\end{aligned} \tag{7}$$

This is a system of two coupled periodically driven oscillators. It is important that the driving in both systems is not identical, but comes with the phase shift (6). If we neglect for a moment this phase shift, the system (7) becomes a system of coupled *identical* chaotic oscillators, with a transition to *complete* synchronization to be observed [6,7]. In the system (7) this happens for $\varepsilon = 0.095$, to be compared with $\varepsilon_l = 0.14$ in the full system. With the phase shift, the transition to lag synchronization occurs. Indeed, if we introduce the lag variables for the second system $\tilde{A}_2 = A_2(t + \tau_0)$, $\tilde{z}_2 = z_2(t + \tau_0)$, where $\tau_0 = (\theta_1 - \theta_2)\omega_0^{-1}$, we can reduce (7) to the system of two identical oscillators, driven with the same force but where the coupling term contains one amplitude that is time shifted. Because the amplitudes in this model are slow, this time shift does not influence the full synchronization significantly, so we get $A_1 \approx \tilde{A}_2, z_1 \approx \tilde{z}_2$. In the initial variables this means the onset of lag synchronization:

$$x_2(t + \tau_0) \approx x_1(t), \quad y_2(t + \tau_0) \approx y_1(t),$$
$$z_2(t + \tau_0) \approx z_1(t).$$

This consideration also explains the discrepancy between the transition point to lag synchronization at $\varepsilon = \varepsilon_l \approx 0.14$ and the point where the second Lyapunov exponent becomes negative ($\varepsilon \approx 0.11$). Indeed, it is known that the transition to complete synchronization is extremely sensitive to small perturbations. Even when the second LE is negative, the local instability can lead to bursts of nonsynchronous behavior [19], see Fig. 4. Because of this intermittency, σ gradually decreases in the region $0.11 < \varepsilon < 0.14$ until these local instabilities disappear.

We now discuss the relation between the lag synchronization and the generalized one. The relation $\mathbf{x}_1(t) \approx \mathbf{x}_2(t + \tau_0)$ can be rewritten as $\mathbf{x}_1(t) \approx T^{\tau}\mathbf{x}_2(t)$, where T^t is the generating operator of the flow of the dynamical system. If the coupling ε and the time lag τ are small, we can approximate T with the generating operator of a partial (uncoupled) Rössler flow; it can be considered as a function in the three-dimensional phase space. Thus, the lag synchronization is similar to GS with the function being defined by the dynamics of the partial system.

To check the universal character of the LS, we investigate numerically two dynamical models of real physical systems. One is the electronic circuit experimentally studied in [3] in the context of CS; the other is the hybrid laser system experimentally studied in [20]. Both systems are

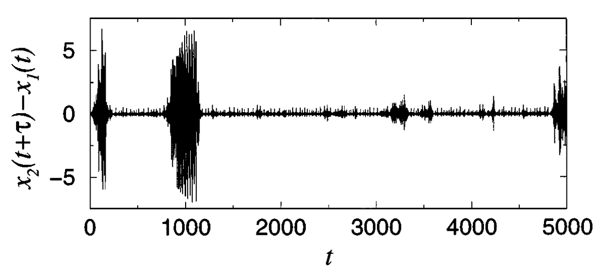

FIG. 4. The time series $x_2(t + \tau) - x_1(t)$ in the intermittent region $\varepsilon = 0.13$, $\tau = 0.32$. The bursts can be viewed as the excursions from the low-dimensional "synchronous" attractor.

described with low-dimensional models and allow one to implement coupling in a straightforward way. We have observed regimes of chaotic lag synchronization in both cases [21], with the similarity function having a rather sharp minimum. E.g., in coupled circuits [3] the similarity function $S(\tau)$ attains its minimum $\sigma = 0.01$ for $\tau = 0.21$ [to be compared with $S(0) = 0.07$]. For the coupled laser system the LS is even more pronounced: $\sigma = 0.005$ for $\tau = 0.3$, while $S(0) = 0.19$.

In summary, we have studied the synchronization properties of two mutually coupled self-sustained chaotic oscillators and have found a new synchronous state, which we refer to as the lag synchronization. We have shown that with the increase of the coupling strength the system can undergo several transitions. First, phase synchronization appears; by this transition, one of the zero LE becomes negative. Further increase of coupling leads to the occurrence of the relationship between the chaotic amplitudes. As a result, the states of two interacting systems coincide (if shifted in time); in the Lyapunov spectrum this transition corresponds to the zero crossing by one of the positive LEs. The motion in the originally six-dimensional phase space is now confined to a nearly three-dimensional manifold, thus corresponding to characterization of a synchronous regime via attractor dimensions [22]. Further increase of coupling decreases the time shift τ_0, and the systems tend to be completely synchronized. We emphasize that, in the LS state, full coherence of *nonidentical* systems is achieved due to interaction. This may be important, e.g., for coherent summation of radiation in laser arrays. As real systems can be hardly found fully identical, the LS can be more frequently encountered in experiments with coupled systems than CS. Finally, with the help of LS we can consider synchronization of periodic and chaotic oscillators within a common theoretical framework. Indeed, due to phase shift in the synchronous state, mutual entrainment of periodic oscillators having different frequencies can be viewed as a particular case of lag synchronization, but not of the complete one.

We thank C. Grebogi, U. Feudel, L. Kocarev, U. Parlitz, and M. Zaks for valuable discussions. M.R. and A.P. acknowledge support from the Max Planck Society.

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