Dynamical Localization in the Paul Trap

M. El Ghafar, P. Törmä, V. Savichev, E. Mayr, A. Zeiler, and W. P. Schleich
Abteilung für Quantenphysik, Universität Ulm, 89069 Ulm, Germany
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We show that quantum localization occurs in the center-of-mass motion of an ion stored in a Paul trap and interacting with a standing laser field. The present state-of-the-art ion trap systems makes the experimental observation of this phenomenon feasible. [S0031-9007(97)03219-5]

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The phenomenon of dynamical localization—an analog of Anderson localization [1] of electronic waves in one-dimensional disordered solids—is a fingerprint of quantum chaos [2]. This has motivated the experimental verification of the suppression of ionization of Rydberg atoms in microwave fields [3] and the localization in the momentum distribution of an atom moving in a phase modulated standing wave [4,5]. In this Letter we show that the effect of dynamical localization of quantum mechanical wave packets appears also in the center-of-mass motion of a single ion confined in a Paul trap [6] and interacting with a laser field. This system is of particular interest, since here the spatial periodicity of the standing wave is broken by the binding potential of the trap. As a result we find localization both in the momentum and the position variables in contrast to the previous examples. The localization lengths are shown to be related to classical diffusion in the energy of a reference oscillator. The recent experimental successes [7] in controlling the quantum motion in a Paul trap make our proposal experimentally feasible.

The phenomenon of dynamical localization in the Paul trap emerges because (i) the Paul trap is an explicitly time dependent device, (ii) a standing laser wave provides a spatially periodic light potential for the center-of-mass motion, and (iii) the temperature of the ion is so low that its motion has to be treated quantum mechanically [8]. Starting from the Hamiltonian describing the motion in the Paul trap in the presence of the standing wave we compute the position and momentum distributions of the ion by solving the corresponding Newton's equations and the Schrödinger equation. We show that the classical distributions are broad [9]. In contrast, the quantum distributions display on top of a broad background a narrow three-peaked distribution. We discuss the relation between the classical diffusion and the quantum localization lengths, and calculate the Floquet states of this system. We conclude by discussing experimental possibilities for observing this phenomenon.

We consider the standard Paul trap setup realized experimentally in many labs [7,10,11]: a standing laser field of frequency ω_L and wave vector k aligned along the x axis couples the internal states of a single two-level ion of mass m to the center-of-mass motion. The resulting dynamics of the ion follows from the time dependent Schrödinger equation with the Hamiltonian $\hat{\tilde{H}} = \hat{\tilde{p}}^2/2m + \frac{1}{2}(m\omega^2/4)[a + 2q\cos(\omega\tilde{t})]\hat{\tilde{x}}^2 + \frac{1}{2}\hbar\omega_a\hat{\sigma}_z + \hbar\Omega_0\hat{\sigma}_x\cos(k\hat{\tilde{x}} + \phi)\cos(\omega_L\tilde{t})$. Here the parameters a and q denote [6] the dc and ac voltages applied to the trap. The frequency of the ac field is ω. Here ω_a is the atomic transition frequency, Ω_0 is the Rabi frequency, and ϕ is the phase of the standing wave.

The phenomenon of dynamical localization is a quantum coherence effect [2]. It is therefore extremely sensitive [12] to noise such as spontaneous emission. In order to avoid spontaneous emission we consider the ion to be initially in its internal ground state and the detuning of the laser field $\Delta = \omega_L - \omega_a$ to be large [13]. After making the rotating wave approximation and introducing the dimensionless position $x \equiv 2k\tilde{x}$, time $t \equiv \omega\tilde{t}/2$, and momentum $p \equiv (4k/m\omega)\tilde{p}$ the dimensionless Hamiltonian

$$\hat{H} \equiv \frac{16k^2}{m\omega^2}\hat{\tilde{H}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}(a + 2q\cos 2t)\hat{x}^2 + \Omega\cos(\hat{x} + 2\phi)$$

with the effective coupling constant $\Omega = 2\hbar k^2\Omega_0^2/m\omega^2\Delta$ governs via the Schrödinger equation $i\bar{k}(\partial/\partial t)|\psi(x,t)\rangle = \hat{H}|\psi(x,t)\rangle$ the vibratory motion of the ion described by the state $|\psi(x,t)\rangle$. Here $\bar{k} = 8k^2\hbar/m\omega$ is the effective Planck constant.

Dynamical localization arises from the properties of the quantum evolution in the domain of classically chaotic dynamics. Choosing the maximum of the cosine potential to be located at the center of the trap potential, i.e., $\phi = 0$ when $\Delta > 0$ or $\phi = \pi/2$ when $\Delta < 0$, the dynamics become chaotic. Indeed in Fig. 1, where we plot the Poincaré surface of section, we observe a chaotic sea with two stable islands in the neighborhood of the minima of the standing laser field at $x = \pm\pi$. These regular structures are remnants of the integrable cases $\Omega = 0$ corresponding to the Mathieu equation and the driven pendulum when $a = q = 0$.

We now calculate the time evolution of a Gaussian wave packet centered initially at the stochastic region near the origin using the split-operator method [14]. For the simulations presented in this Letter the numerical value of the

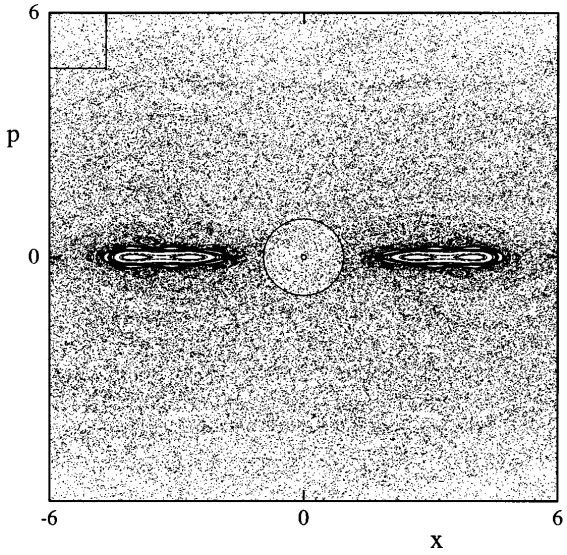

FIG. 1. Poincaré surface of section for an ion moving in a strongly detuned standing wave laser field and a time dependent harmonic potential. The dynamics is chaotic all over the phase space except for two small stable islands around the phase space points $(x = \pi, p = 0)$ and $(x = -\pi, p = 0)$. The circle with its center at the origin of phase space denotes the contour line of exponential decay of the Gaussian phase space distribution of the initial wave packet. The box in the upper left corner shows the area of $2\pi\bar{k}$. Here we have used the trap parameters $a = 0.0$, $q = 0.4$, and the coupling $\Omega = 0.65$.

Planck constant is $\bar{k} = 0.29$. The spatial size of the quantized grid is taken to be $-80 \le x \le 80$, with 4096 grid points. This allows us to resolve momenta up to $p \simeq 23$, which is sufficient for our purpose. To make a comparison to the classical case we calculate 4096 trajectories starting from a classical Gaussian ensemble centered initially at the origin and having the same widths in the position and momentum as the quantum wave packet.

In the top of Fig. 2 we show for an example case the spreads Δx and Δp in position and momentum of the classical and quantum mechanical distributions as functions of time. In order to remove the fast oscillations we have averaged the spreads over one cycle of the rf field. There are two main stages in the time dependence of the momentum and position spreads. In the short time behavior, that is, before the quantum break time $t_* \sim 50$, there is no significant difference between the classical (upper line) and the quantum mechanical (lower line) spreads. In the second stage, which characterizes the long time behavior, that is, for $t > 50$, there is a considerable difference between the classical and the quantum mechanical spreads: whereas the classical ones increase monotonically, the corresponding quantum mechanical ones oscillate with a small amplitude around an average value. This is the first indication that the quantum mechanical distributions show dynamical localization.

To bring this out most clearly we show in the lower part of Fig. 2 the time averaged probability distributions of position and momentum. The classical distributions are broad, while the quantum mechanical ones are dynamically localized. The mixed phase space with stochastic domains and stable islands results from the combination of the trap potential and the standing wave, and reflects itself in the three-peak structure of the position distribution. Note that classically these peaks are missing. We have also verified that dynamical localization is not sensitive to the initial position of the wave packet in the chaotic region and hence not destroyed by small fluctuations in the optical phase ϕ.

A natural explanation of our results emerges when we make a time transformation which expresses the system as a combination of a reference harmonic oscillator and a time dependent potential. We set $\tau(t) = \int_0^t dt'\, r(t')^{-2}$, where $r(t) = |\epsilon(t)|$ is the modulus of the solution $\epsilon = \epsilon(t)$ of the classical Mathieu equation. In the new position variable $\chi(\tau) = r(t)^{-1}x(t)$ and momentum variable $\Pi \equiv d\chi(\tau)/d\tau$ the Hamiltonian reads $H = \Pi^2/2 + \omega_r^2\chi^2/2 + \Omega r(t)^2 \cos[r(t)\chi]$, where ω_r is the frequency of the time independent reference oscillator. The dynamics simplifies considerably when we choose the reference oscillator [15] $\omega_r = \omega_s + \sum_{-\infty}^{\infty} n c_n$, where ω_s is the secular frequency and c_n are the Fourier coefficients of $\epsilon(t)$. With $\epsilon(0) = 1$, $\dot{\epsilon}(0) = i\omega_r$ as the choice of initial conditions $\epsilon(t)$ becomes the Floquet solution. Then $r(t)$ is π periodic and there is a simple one to one correspondence between the new and the old variables.

When we now introduce the action-angle variables I and θ for a harmonic oscillator of frequency ω_r by $\chi \equiv \sqrt{2I/\omega_r}\sin\theta$ and $\Pi \equiv \sqrt{2I\omega_r}\cos\theta$, the Hamiltonian of our system reduces to $H_I = \omega_r I + \Omega V(I, \theta, t)$, where $V(I, \theta, t) = r(t)^2 \cos[r(t)\sqrt{2I/\omega_r}\sin\theta]$. Hence the potential V causes quasirandom behavior of θ and slow diffusion in I [16]. If we approximate the diffusion coefficient [16] $D(I) = \Omega^2/[2\pi\tau(\pi)]\int_0^{2\pi} d\theta[\int_0^{\pi} dt \times \partial V(I, \theta, t)/\partial\theta]^2$ by a constant $D(I) \sim D$, then normal linear diffusion of I leads to anomalous diffusion [17] with $\Delta\chi, \Delta\pi \propto t^{1/2}$. Furthermore, $D_\chi \sim \sqrt{D}/\omega_r$ and $D_\Pi \sim \sqrt{D}\,\omega_r$ implies $D_\Pi/D_\chi = \omega_r^2$. Diffusion in x and p should be similar when averaging over one period of $r(t) = r(t + \pi)$.

We have tested this hypothesis numerically by carrying out simulations for $q = 0.2$, 0.3, and 0.4 and $\Omega \in [0.4, 1.3]$. Indeed $\Delta x^2, \Delta p^2 \sim t^\alpha$, where α varies in the range $[0.47, 0.74]$, and $D_p/D_x = \nu^2$ is a constant (mean deviation 1%) for a given q, which is in accordance with the fact that ω_r depends only on q. Furthermore, ν is close [18] to ω_r, whereas it deviates considerably from the secular frequency ω_s describing the time averaged binding potential of the Paul trap.

We can relate the widths of the localized quantum wave packet to the classical diffusion constants D_x, D_p, and the break time t_* by $\Delta x^2 = D_x t_*^\alpha$ and $\Delta p^2 = D_p t_*^\alpha$. We have also confirmed that $\Delta p^2/\Delta x^2 = \nu^2$, with a mean deviation of 10% originating from the oscillation of the widths; see Fig. 2. Therefore we now consider the

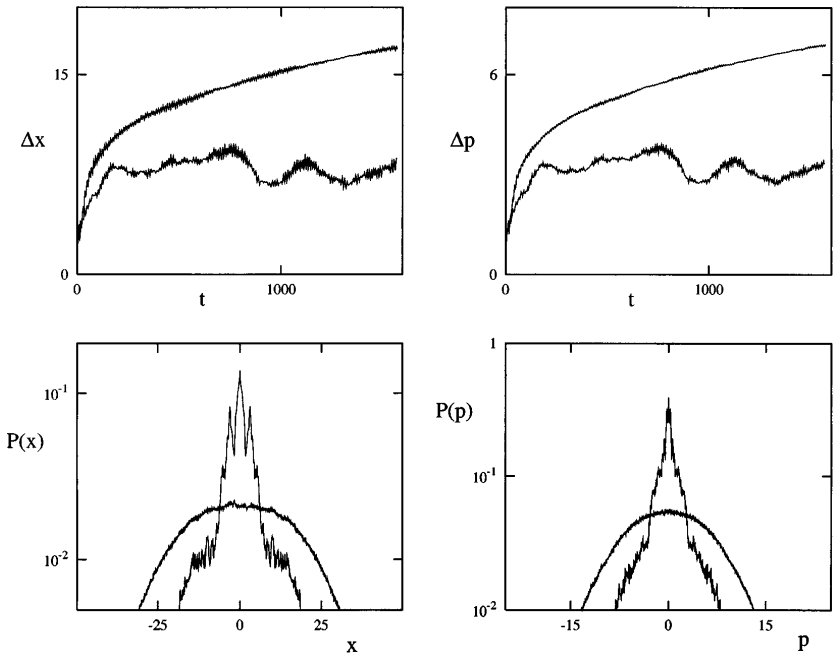

FIG. 2. Classical and quantum dynamics of a single ion moving under the influence of a strongly detuned standing laser field and a time dependent harmonic potential. On the top we show the time dependence of the widths of the classical (upper curve) and quantum mechanical (lower curve) position (left) and momentum (right) distributions. We find that classically both widths increase with time, whereas in the quantum case they oscillate around an average value. On the bottom we show in a semilogarithmic plot the corresponding position (left) and momentum (right) distributions averaged over time in an interval of $\Delta t = 50\pi$ around $t = 475\pi$. Indeed the classical distributions are broad giving rise to a polynomial curve, whereas the quantum ones consist of narrow distributions which rest on a broad pedestal. Note that the $P(x)$ axis does not begin from zero, so a minor part of the distributions is left outside the figure. Here we have used a wave packet of width $\sigma_{0x}^2 = \hbar = 0.29$ and the trap parameters $a = 0.0$, $q = 0.4$, with the coupling $\Omega = 0.65$.

localization length l in the reference oscillator basis with the frequency ν, i.e., $\Delta x^2 = \hbar/\nu(l + \frac{1}{2})$ and $\Delta p^2 = \hbar\nu(l + \frac{1}{2})$. In Fig. 3 we show l and t_* as a function of Ω for several q. We note that $l \sim t_*$, which enables us to write the closed expression $l^{1-\alpha} \sim D_p/(\hbar\nu) = D_x\nu/\hbar$. For $\alpha \sim 1/2$ we have $l \sim \mathcal{D}/\hbar^2$ where $\mathcal{D}$ is the diffusion constant for the energy of the reference oscillator. This relation is of the same form as the one derived for the localization in momentum of the kicked rotator [2].

In order to gain deeper insight into this phenomenon of localization we now calculate numerically the Floquet quasienergies μ_k and the corresponding eigenstates $\psi_k(x,t) = e^{-i\mu_k t}u_k(x,t)$ of the trapped ion–laser field system. Here the functions $u_k(x,t)$ are periodic in time with period π. We can obtain the Floquet states as the eigenstates of the eigenvalue equation $U(t + \pi, t)\psi_k(x,t) = e^{-i\mu_k\pi}\psi_k(x,t)$ with the time-evolution operator $U(t + \pi, t)$ propagating the state over one time period. We construct $U(t + \pi, t)$ by integrating the Schrödinger equation for the 200 lowest eigenstates of a stationary reference oscillator [15].

We calculated the quasienergies as a function of the coupling. The two lowest quasienergies start to grow, and become almost degenerate for couplings $\Omega \geq 0.2$. This behavior results from an effective double-well potential caused by the standing laser field and the trap potential [19]. The existence of nearly degenerate doublets is connected [20] to the possibility of quantum tunneling between the stable islands [21]—a topic discussed in more detail in an upcoming paper.

The Floquet solutions ψ_k form a complete orthogonal basis. Hence we can represent any wave packet solution $\psi(x,t)$ of the Schrödinger equation as a superposition $\psi(x,t) = \sum_k A_k\psi_k(x,t)$ with time independent coefficients A_k. In this respect Floquet states play a role similar to that of the energy eigenstates in the time independent case, and the quantities μ_k have the meaning of time averaged energies. We have calculated A_k for several parameters and found that the distribution is strongly localized; for instance, for the parameters of Fig. 2 only four states are enough to cover 60% of the initial wave packet. We now expand $\psi(x,t) = \sum_k A_k e^{-i\mu_k t}\sum_n a_n^k(t)\phi_n(x)$, where $\phi_n(x)$ are the eigenstates of the reference oscillator and $a_n^k(t + \pi) = a_n^k(t)$. In the inset of Fig. 3 we show the time averaged distribution $|\psi(n)| = [\sum_k |A_k|^2|a_n^k(0)|^2]^{1/2}$ of the localized wave packet in the basis of the quantum number n of the reference oscillator, and compare it to the distribution $\mathcal{N}e^{-n/l}$, where l is the localization length in n obtained from the wave packet simulations. We find good agreement with the predicted localization length. Hence we have shown that dynamical localization in our

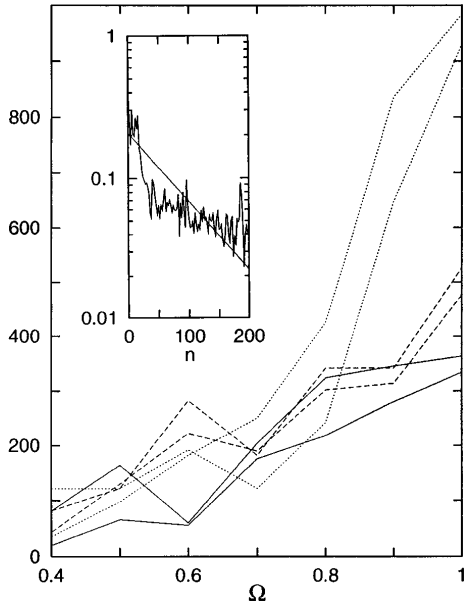

FIG. 3. The localization length l and the break time t_* as a function of the coupling Ω for $q = 0.2$, 0.3, and 0.4 (dotted, dashed, and solid lines, respectively). Starting from $\Omega = 0.4$, l is the lower curve. The inset shows in log-scale the time averaged distribution $|\psi(n)|$ of the localized wave packet in the reference oscillator basis, and the distribution $\mathcal{N}\exp(-n/l)$, where $l = 90$ is predicted by the wave packet simulations and $\mathcal{N} = 0.2$ is the normalization. The parameters in the inset are the same as in Fig. 2.

system can be described as inhibition of classical diffusion in the action variable (vibrational quantum number) of a Floquet reference oscillator.

The observation of this phenomenon is possible with the present ion trap systems. Indeed taking the experimental trap parameters from [7] and considering a dipole transition from the ground state of $^9Be^+$ we obtain for a driving frequency $\omega/2\pi \sim 200$ MHz the values $q \sim 0.2$ and $\bar{k} \sim 0.015$. In order to achieve the value $\bar{k} \sim 0.3$ used in our simulations we need a smaller frequency such as $\omega/2\pi \sim 10$ MHz; to keep q in the stable region the applied voltage has to be [6] smaller, or the trap size larger, than in [7]. To be consistent with the assumption of far detuning, the term $\Omega_0/\Delta \equiv \epsilon$ in the dimensionless coupling $\Omega = (\bar{k}\Delta)/(4\omega)\epsilon^2$ has to be small. For $\epsilon = 0.1$ and the detuning $\Delta/2\pi \sim 10$ GHz we obtain $\Omega = 0.65$, as used in our simulations. Moreover Fig. 3 shows that localization occurs for a wide range of the parameters.

We conclude by summarizing our main results. The motion of a trapped ion interacting with a laser field shows the phenomenon of dynamical localization in position and momentum. This phenomenon can be observed by measuring the fluorescence light from the ion—using this technique the measurement of a position distribution is possible either directly [4,10] or via the vibrational state distribution [7,11]. Moreover the three-peak stucture of the quantum mechanical position distribution, reflecting the mixed phase space, opens up new possibilities to study the phenomenon of dynamical quantum tunneling.

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