Universality of Low Energy Absorption Cross Sections for Black Holes

Sumit R. Das,[1] Gary Gibbons,[2] and Samir D. Mathur[3]
[1]Tata Institute of Fundamantal Research, Homi Bhabha Road, Mumbai 400 005, India
[2]DAMTP, Cambridge University, Silver Street, Cambridge CB3 9EW, United Kingdom
[3]Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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In this paper we compute the low energy absorption cross section for minimally coupled massles scalars and spin-1/2 particles into a general spherically symmetric black hole in arbitrary dimensions. The scalars have a cross section equal to the area of the black hole, while the spin-1/2 particles give the area measured in a flat spatial metric conformally related to the true metric. [S0031-9007(96)02146-1]

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Recently there has been great interest in the possibility of relating some of the properties of classical black hole solutions of the low energy supergravity [1] limits of string theories to a more fundamental microscopic description based on strings and *D*-branes. In particular extremal black holes correspond in many cases to BPS (Bogomolny-Prasad-Sommerfeld) states of the theory, and their number for a given set of charges is expected to be independent of coupling. This allows a comparison between the number of particle states (computable at weak coupling) with the Bekenstein-Hawking entropy of the black hole (which would exist for the same charges at strong coupling). Agreement is found in all the cases investigated so far [2], thus suggesting that the Bekenstein-Hawking entropy for a hole does indeed correspond to the count of microstates for the hole, though it is still unclear where these microstates actually reside.

To study interesting processes like Hawking radiation, we need to allow quanta to fall into the hole, rendering it non-BPS, after which it would evaporate back towards extremality. Are there relations between the properties of particle states at weak coupling and properties of black holes, when we consider deviations from extremality? One result in this direction was presented in [3] where it was shown that if one naively ignores interaction between non-BPS states, then the degeneracy of a collection of branes and antibranes continues to reproduce the Bekenstein-Hawking entropy for nonextremal holes and leads to the correct Hawking temperature. In [4,5] it was found that if one computes the low energy cross section for absorption and emission of neutral scalars in the $4 + 1$ dimensional extremal black hole, then this cross section agrees exactly with that for absorption or emission with the corresponding collection of branes at weak coupling. This result has been extended to charged scalars in four and five dimensions in [6]. Recently it has been shown [7] that the *D*-brane decay reproduces the correct grey body factors both for neutral and charged scalar emission.

To discover if these are examples of a general pattern of universality in the theory, we need to observe universalities that may exist in the interactions of classical black holes. The absorption cross section for low energy particles in $3 + 1$ dimensional black holes was studied extensively in the past, for example, by Starobinski and Churilov [8], Gibbons [9], Page [10], and Unruh [11]. In these calculations if we consider the particle to be a massless minimally coupled scalar, then we find that the cross section equals the area of the horizon of the black hole. In [12] and [4], the $4 + 1$ dimensional cases studied also yielded a cross section equal to the area of the horizon. It was found in [13], however, that the low energy cross section for *fixed* scalars (i.e., scalars which take fixed values at the horizon of some extreme black holes) is suppressed by powers of the frequency.

In this paper we show that for all spherically symmetric black holes, regardless of the theory in which they arise, the low energy cross section for massless minimally coupled scalars is always the area of the horizon. Further, we also find the corresponding result for minimally coupled massless spin-1/2 quanta, and this also exhibits a universal form. Note that the absorption processes studied here are not for black holes close to extremality though they do have the restriction to low energies.

We will consider general metrics in $(p + 2)$ spacetime dimensions of the form

$$ds^2 = -f(r)dt^2 + g(r)[dr^2 + r^2 d\Omega_p^2], \qquad (1)$$

where $d\Omega_p$ is the metric on the unit p sphere. The functions $f(r)$ and $g(r)$ are chosen to ensure that the metric is asymptotically flat.

At low energies only the mode with lowest angular momentum will contribute to the cross section. For scalars this is the s wave. The mode $\phi_\omega(r)$ with frequency ω satisfies the equation

$$\{(r^p[f(r)]^{\frac{1}{2}}[g(r)]^{\frac{p-1}{2}}\partial_r)^2 + \omega^2[r^2 g(r)]^p\}\phi_\omega(r) = 0\,. \qquad (2)$$

Define a coordinate τ by the relation

$$d\tau = \frac{dr}{r^p[f(r)]^{\frac{1}{2}}[g(r)]^{\frac{p-1}{2}}}\,, \qquad (3)$$

so that (2) becomes

$$\{\partial_\tau^2 + \omega^2[r^2 g(r)]^p\}\phi_\omega(\tau) = 0\,. \qquad (4)$$

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Let the horizon be at the position $r = r_H$. The area of the horizon is

$$A_H = \{r_H[g(r_H)]^{\frac{1}{2}}\}^p \omega_p \equiv R_H^p \omega_p \,, \tag{5}$$

where Ω_p is the volume of the unit p sphere.

In the following we will restrict our considerations to the lowest order in $\lambda = \omega l$, where l is the largest length scale in the solution (1).

Close to the horizon we can write the solution of (4) by treating $r^2 g(r) \sim R_H^2$ to be a constant. This is the near region $\omega r < \lambda$. At the horizon we want a purely ingoing wave, which is given by $\phi_\omega(\tau) = e^{-i\omega R_H^p \tau}$. At distances $\omega R_H^p \tau \ll 1$ but $\omega r < \lambda$ this solution behaves as

$$\phi_\omega \sim 1 - i\omega R_H^p \tau, \qquad \tau \sim -\frac{1}{p-1} r^{-(p-1)}. \tag{6}$$

This is justified since in this region $\omega R_H^p \tau \ll 1$ means $\omega r \gg (\omega R_H)^{\frac{p}{p-1}}$ which is consistent with the condition $\omega r < \lambda$ since we have $\omega R_H < \lambda \ll 1$.

In the region far from the horizon, $\omega r > \lambda$ the wave equation approximates to that in a flat metric

$$\left[\partial_\rho^2 - \frac{p(p-2)}{4\rho^2} + 1\right][\rho^{p/2}\phi_\omega(\rho)] = 0\,, \tag{7}$$

where we have used rescaled variables $\rho = \omega r$. The corrections to Eq. (7) involve higher powers of λ. This may be seen by considering the exact equation and noting that when expressed in terms of ρ the only dependences on ω are contained in the functions $f(r)$ and $g(r)$. Let us expand

$$f(r) = 1 + \sum_{n=0}^{\infty}\left[\frac{\omega f_n}{\rho}\right]^{p-1+n} \tag{8}$$

and similarly for $g(r)$. The coefficients f_n's are various length scales associated with the solution. The leading power in (8) is determined by the fact that in the asymptotic region one must have Coulomb law behavior. Thus the terms in the far region equation which involve departures of $f(r)$ and $g(r)$ from unity are all suppressed by powers of λ.

The solution of (7) is

$$\phi_\omega(\rho) = \rho^{\frac{1-p}{2}}[AJ_{(\frac{p-1}{2})}(\rho) + BJ_{-(\frac{p-1}{2})}(\rho)]. \tag{9}$$

For $r\omega \ll 1$ (but still $\omega r > \lambda$) this reduces to

$$\phi_\omega(\rho) \sim \frac{2^{-(\frac{p-1}{2})}A}{\Gamma(\frac{p+1}{2})} + \frac{2^{(\frac{p-1}{2})}\omega^{1-p}}{\Gamma(\frac{3-p}{2})}\frac{B}{r^{p-1}}\,, \tag{10}$$

whereas for $r\omega \gg 1$ this becomes

$$\phi_\omega \sim \sqrt{\frac{2}{\pi\rho^p}}\left\{A\cos\left[\rho - \frac{\pi(p-1)}{4} - \frac{\pi}{4}\right] + B\cos\left[\rho + \frac{\pi(p-1)}{4} - \frac{\pi}{4}\right]\right\}, \tag{11}$$

where for p an odd integer, we take the analytic continuation of all expressions in p. Matching onto (6) in the overlap region we find

$$\frac{B}{A} = i\frac{2^{-(p-1)}(\omega R_H)^p}{p-1}\frac{\Gamma(\frac{3-p}{2})}{\Gamma(\frac{p+1}{2})}\,, \tag{12}$$

which gives for the absorption probability of an $l = 0$ spherical wave

$$\Gamma = 1 - \left|\frac{1 + \frac{B}{A}e^{i\pi(p-1)/2}}{1 + \frac{B}{A}e^{-i\pi(p-1)/2}}\right|^2 \tag{13}$$

$$= 4\frac{2^{-(p-1)}}{p-1}(\omega R_H)^p \sin[\pi(p-1)/2]\frac{\Gamma(\frac{3-p}{2})}{\Gamma(\frac{1+p}{2})} \tag{14}$$

in the limit $\omega \to 0$. To convert the spherical wave absorption probability into the absorption cross section we have to extract the ingoing s wave from the plane wave:

$$e^{ikz} \to e^{-ikr}r^{-p/2}Y_{00}K\,, \tag{15}$$

where $Y_{00} = \Omega_p^{-1/2}$ is the normalized s wave function on the p sphere. Using $\Omega_p = \frac{2\pi^{\frac{p+1}{2}}}{\Gamma(\frac{p+1}{2})}$ we get

$$|K|^2 = \frac{1}{4\omega^p}\Omega_p^{-1}\Omega_{p-1}^2 2^p(\Gamma[p/2])^2, \tag{16}$$

so that the absorption cross section σ becomes

$$\sigma = \Gamma|K|^2 = \frac{2\pi^{(p+1)/2}R_H^p}{\Gamma[(p+1)/2]} = A_H\,, \tag{17}$$

where A_H is the area of the horizon.

For minimally coupled massless spinors, the Dirac equation may be written down by making use of the properties of the massless Dirac operator under conformal transformations (see, e.g., [14]):

$$\nabla_\mu\gamma^\mu\psi = f^{-\frac{1}{2}}\gamma^0\partial_0[\psi] + (fg^{p+2})^{-\frac{1}{4}}\gamma^i\partial_i[g^{\frac{p}{4}}f^{\frac{1}{4}}\psi] = 0\,. \tag{18}$$

Define $\chi = f^{1/4}g^{p/4}\psi$ and $h = \sqrt{f/g}$. Then the equation is

$$h\gamma^i\partial_i\chi = i\omega\gamma^0\chi\,. \tag{19}$$

Note that

$$\gamma^i\partial_i = \gamma^r\left[\partial_r + \frac{p}{2r}\right] + \frac{1}{r}(\gamma^i\nabla_i)_T\,, \tag{20}$$

where the subsctipt T stands for the part of the differential operator tangent to the p sphere. Write

$$\chi = \sum_{n=0}^{\infty}F_n(r)\lambda_n^+ + G_n(r)\lambda_n^-, \tag{21}$$

where $\lambda^\pm$ are mutually orthogonal functions of the angular coordinates only. They satsify

$$\begin{aligned}\gamma^r\gamma^0\lambda_n^\pm &= \lambda_n^\mp \\ \gamma^r(\gamma^i\nabla_i)_T\lambda_n^\pm &= \mp\left(n + \frac{p}{2}\right)\lambda_n^\pm.\end{aligned} \tag{22}$$

Then we get

$$h\left\{\lambda_n^-\left(\partial_r + \frac{p}{2r}\right)F_n + \lambda_n^+\left(\partial_r + \frac{p}{2r}\right)G_n + \frac{1}{r}\left[-\left(n + \frac{p}{2}\right)F_n\lambda_n^- + \left(n + \frac{p}{2}\right)G_n\lambda_n^+\right]\right\} = i\omega[F_n\lambda_n^+ + G_n\lambda_n^-]. \quad (23)$$

Setting to zero the coefficients of $\lambda^\pm$ we get

$$h\left[\partial_r G_n + (p + n)\frac{G_n}{r}\right] = i\omega F_n\,,$$
$$h\left[\partial_r F_n - n\frac{F_n}{r}\right] = i\omega G_n\,. \quad (24)$$

The lowest angular momentum modes are found for $n = 0$, which gives (with $F \equiv F_0$)

$$\partial_r^2 F + \partial_r F\left[\frac{\partial_r h}{h} + \frac{p}{r}\right] + \omega^2 h^{-2}F = 0\,. \quad (25)$$

Define the new coordinate x through

$$\frac{d}{dx} = [h(r)r^p]\frac{d}{dr}\,. \quad (26)$$

The equation becomes

$$\partial_x^2 F + \omega^2 r^{2p}F = 0\,. \quad (27)$$

Again choosing an ingoing wave at the horizon, the analog of (6) is

$$F = 1 - i\omega R_H^p g_H^{-(d-1)/2}\frac{r^{-(p-1)}}{p - 1}\,, \quad (28)$$

where $g_H = g(r_H)$ is the value of $g(r)$ at the horizon. In (28) we have used (26) to solve for x for large r

$$x = \frac{\rho^{-p+1}}{(-p + 1)}\,. \quad (29)$$

Comparing to the case of the scalar, we see that the absorptionprobablity is $g_H^{-p/2}$ times the result for the scalar.

It is interesting to note for extremal holes $r_H \to 0$ and $g_H \to \infty$ so that the absorption cross section for minimally coupled fermions vanishes in this limit of extremality.

The absorption probablility above implies a cross section

$$\sigma = 2g_H^{-p/2}A_H\,. \quad (30)$$

Here the factor of 2 comes from the two spinors $\lambda^\pm$ that contribute to the absorption at low energies, when the incident wave is a plane wave times a constant spinor.

Note that (30) is $2\omega_p r^p$, where $\omega_p r^p$ is the area of the horizon measured in the spatial metric $ds^2 = dr^2 + r^2 d\Omega^2$, which is conformal to the spatial metric in (1). Here r is the isotropic radial coordinate.

It is well known that $N = 2$ supergravity has black hole solutions that preserve supersymmetry. In that case we expect that the cross section for the scalars and for their spinor superpartners are related. The equation for these spinors is not, however, the minimal Dirac equation, since the superpartners of minimally coupled scalars have a coupling to the electromagnetic field strength. This case is under investigation.

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