

Scaling Hypothesis for the Spectral Densities in the O(3) Nonlinear Sigma Model

J. Balog* and M. Niedermaier

Max-Planck-Institut für Physik, Werner Heisenberg Institut, Föhringer Ring 6, 80805 Munich, Germany

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A scaling hypothesis for the n -particle spectral densities of the O(3) nonlinear sigma model is described. It states that for large n the n -particle spectral densities are “self-similar” in being rescaled copies of a universal shape function. Promoted to a working hypothesis, it allows one to compute the two-point functions at all energy scales and yields exact values for two nonperturbative constants. Applied to nonintegrable perturbations of the model, the hypothesis implies scaling laws for multiparticle production processes analogous to Koba-Nielsen-Olesen scaling in QCD. [S0031-9007(97)03266-3]

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An efficient way to describe the two-point function of some local operator \mathcal{O} in a relativistic QFT is in terms of a Källén-Lehmann spectral representation. The spectral density $\rho(\mu)$ of \mathcal{O} can be viewed as a measure for the number of degrees of freedom coupling to \mathcal{O} at energy μ . It decomposes into a sum of n -particle contributions

$$\rho(\mu) = \sum_n \rho^{(n)}(\mu), \quad (1)$$

where, depending on the local operator under consideration, some of the n -particle contributions may vanish on the grounds of internal quantum numbers. In a theory with a single mass scale m one has $\rho^{(1)}(\mu) \sim \delta(\mu - m)$ and $\rho^{(n)}(\mu)$, $n \geq 2$ has support only above nm , i.e., above the n -particle threshold. Once $\rho(\mu)$ is known, the various (Minkowski space or Euclidean) two-point functions of \mathcal{O} can be computed as convolution integrals with an appropriate kernel carrying only kinematical information. The dynamical problem consists in computing the n -particle spectral densities $\rho^{(n)}(\mu)$ of \mathcal{O} .

Here we shall be concerned with massive 1 + 1 dimensional QFTs and in particular with the O(3) nonlinear sigma (NLS) model. The four most interesting local operators in this model are the spin field, the Noether current, the energy momentum (EM) tensor, and the topological charge (TC) density. Their spectral densities can be grouped into two families $\rho_l^{(n)}(\mu)$, $n \geq 1$, $l = 0, 1$ according to their isospin. For n even/odd the $\rho_0^{(n)}$ are the spectral densities of the EM tensor/TC density, respectively; similarly, $\rho_1^{(n)}$ for n even/odd are the spectral densities of the current/spin, respectively. The following pieces of information are available for these spectral densities: (i) For small particle numbers n the functions $\rho_l^{(n)}(\mu)$ can be computed *exactly* by means of the form factor approach. In [1] this has been done up to six particles. (ii) For all particle numbers n the $\mu \rightarrow \infty$ asymptotics of the n -particle spectral densities is known, and is given by

$$\rho_l^{(n)}(\mu) \sim \frac{A_l^{(n)}}{\mu(\ln \mu)^{4-2l}}, \quad \mu \rightarrow \infty, \quad (2)$$

where the constants $A_l^{(n)}$ are computable from the integrals of the lower particle spectral densities [1]. The constants $A_l^{(n)}$ are rapidly increasing with n . This implies that the $\mu \rightarrow \infty$ asymptotics of the full spectral densities (1) cannot be computed by naively summing up the asymptotic expressions (2), which in fact would be divergent. (iii) The large μ asymptotics of the full spectral densities can, however, be computed in renormalized perturbation theory (PT). (The correctness of PT in this model has been challenged in [2]. To simplify the exposition we shall assume the validity of PT for the UV asymptotics throughout this Letter.) One finds for the leading behavior

$$\begin{aligned} \text{EM and top: } \rho(\mu) &\sim \frac{A^{\mathcal{O}}}{\mu} \left[\frac{1}{(\ln \mu)^2} + O\left(\frac{\ln \ln \mu}{(\ln \mu)^3}\right) \right], \\ \text{spin and curr: } \rho(\mu) &\sim \frac{A^{\mathcal{O}}}{\mu} \left[1 + O\left(\frac{1}{\ln \mu}\right) \right]. \end{aligned} \quad (3)$$

Subleading terms can also be computed, but not all of the overall constants are accessible to PT. In particular, $\lambda_1 := A^{\text{spin}}$ is an unknown nonperturbative constant. In the case of the TC density A^{top} is fixed by PT but its relation to the nonperturbatively defined spectral sum (1) is not. Equivalently, the matrix elements of the TC density between the vacuum and some multiparticle state are defined only up to an unknown nonperturbative constant λ_0 .

Missing pieces of information about the spectral densities are the following: (iv) One would like to be able to compute the full spectral densities for all $\mu \geq 0$, not only their large μ asymptotics. This would allow one to compute the two-point functions at *all* energy/length scales. In terms of the spectral resolution (1) this amounts to knowing all the n -particle contributions, not only those with $n \leq n_0$ for which the computation can be done explicitly. (v) One would like to know the (exact) values of the nonperturbative constants λ_0 and λ_1 . Knowledge of these constant would allow one to match nonperturbative and perturbative information unambiguously. In this respect, their role is similar to that of the m/Λ ratio [3].

The purpose of this Letter is to bridge the gap between the perturbative and the nonperturbative regime and to provide the missing pieces of information (iv) and (v). It is based on a remarkable self-similarity property of the n -particle spectral densities. For large n they appear to be basically rescaled copies of a “universal shape function” $Y_l(z)$. Explicitly,

$$\rho_l^{(n)}(\mu) \approx \frac{M_l^{(n)}}{\mu} Y_l\left(\frac{\ln(\mu/m)}{\xi_l^{(n)}}\right), \quad l = 0, 1, \quad (4)$$

where $M_l^{(n)}$ and $\xi_l^{(n)}$ are certain scaling parameters to be specified later. In the following we shall first give a precise formulation of the scaling law (4) and recall some of the evidence presented for it in [1]. Then we shall promote it to a working hypothesis and show that it has the following consequences: The UV behavior is consistent with PT; in particular those coefficients A^θ in (3) accessible to PT are reproduced. The nonperturbative constants λ_0 and λ_1 are determined exactly and in the normalization [1] are given by

$$\lambda_0 = \frac{1}{4}, \quad \lambda_1 = \frac{4}{3\pi^2}. \quad (5)$$

Finally, candidate results for the two-point functions at all energy or length scales are obtained.

Formulation of the hypothesis.—With hindsight to the asymptotics (2) let us introduce

$$R_l^{(n)}(x) := me^x \rho_l^{(n)}(me^x), \quad l = 0, 1. \quad (6)$$

Here $l = 0, 1$ as before correspond to the EM tensor and TC density and spin and current series, respectively. The graphs of these functions are roughly “bell shaped”: Starting from zero at $x = \ln n$ they are strictly increasing, reach a single maximum at some $x = \xi_l^{(n)} > \ln n$, and then decrease monotonically for all $x > \xi_l^{(n)}$. The position $\xi_l^{(n)}$ of the maximum and its value $M_l^{(n)} = R_l^{(n)}(\xi_l^{(n)})$ are two important characteristics of the function, and hence of the spectral density. Defining

$$Y_l^{(n)}(z) := \frac{1}{M_l^{(n)}} R_l^{(n)}(\xi_l^{(n)} z), \quad l = 0, 1, \quad (7)$$

both the value and the position of the maximum are normalized to unity. Initially $Y_l^{(n)}(z)$ is defined for $(\ln n)/\xi_l^{(n)} \leq z < \infty$; in order to have a common domain of definition we set $Y_l^{(n)}(z) = 0$ for $0 \leq z \leq (\ln n)/\xi_l^{(n)}$. The proposed behavior of the spectral densities is as in the following *scaling hypothesis*:

(a) (*Self-similarity*) The functions $Y_l^{(n)}(z)$, $n \geq 2$ converge pointwise to a bounded function $Y_l(z)$. The sequence of k th moments converges to the k th moments of $Y_l(z)$ for $k + l = 0, 1$, i.e.,

$$\lim_{n \rightarrow \infty} Y_l^{(n)}(z) = Y_l(z), \quad z \geq 0,$$

$$\lim_{n \rightarrow \infty} \int_0^\infty dz z^k Y_l^{(n)}(z) = \int_0^\infty dz z^k Y_l(z).$$

(b) (*Asymptotic scaling*) The parameters $\xi_l^{(n)}$ and $M_l^{(n)}$ scale asymptotically according to powers of n , i.e.,

$$\xi_l^{(n)} \sim \xi_l n^{1+\alpha_l}, \quad M_l^{(n)} \sim M_l n^{-\gamma_l}.$$

Feature (a) in particular means that for sufficiently large n the graphs of two subsequent members $Y_l^{(n-1)}(z)$ and $Y_l^{(n)}(z)$ should become practically indistinguishable. This appears to be satisfied remarkably well even for small $n = 4, 5, 6$, as is illustrated in Fig. 1 for the $l = 1$ series.

The analysis of part (b) of the scaling hypothesis is more involved. It turns out that all but one of the exponents in (b) are fixed by self-consistency, and only this one has to be determined by fitting against the $n \leq 6$ particle data. The result is [1]

$$\begin{aligned} \gamma_1 = 1, \quad \alpha_0 = \alpha_1 = \alpha, \\ \gamma_0 = 3 + 2\alpha, \quad \alpha \approx 0.273. \end{aligned} \quad (8)$$

Consequences of the hypothesis.—Let n_0 be the maximal particle number for which the spectral densities have been computed explicitly (at present $n_0 = 6$). Then (4) gives candidate expressions for all $n > n_0$ particle spectral densities so that one can evaluate their sum. Concerning the UV behavior of the sum notice that a finite number of terms in the sum, each decaying according to (2), can never produce the different UV behavior in (3). However, the infinite sum does. What is happening is that the partial sums $(\ln \mu)^{2-2l} \sum_{n_0+1}^N \rho_l^{(n)}(\mu)$ develop a plateau, i.e., are practically constant in a large interval $\mu_{\min} \leq \mu \leq \mu_{\max}(N)$. When N is increased the plateau is prolonged and eventually reaches out to infinity, i.e., $\mu_{\max}(N) \rightarrow \infty$ for $N \rightarrow \infty$ (while μ_{\min} is basically N independent). The value of the plateau determines the asymptotic constants in (3). In those cases where the coefficients are accessible to PT the perturbative value is reproduced with an accuracy better than 1%. In addition one obtains

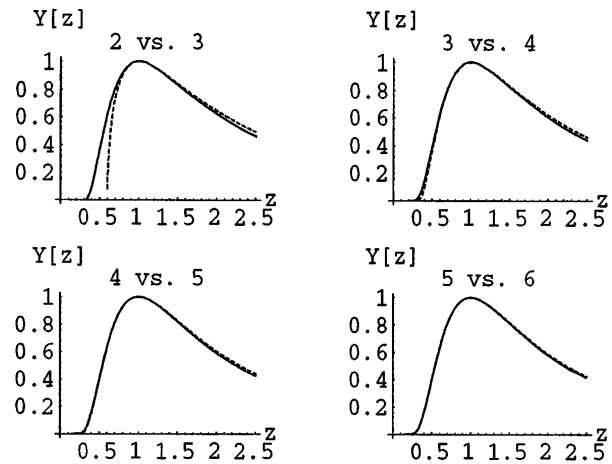


FIG. 1. Illustration of the self-similarity property of the rescaled $l = 1$ spectral densities. The plots show $Y_l^{(n)}(z)$ (dashed) compared with $Y_l^{(n+1)}(z)$ (solid) for $n = 2, 3, 4, 5$.

the following two *exact* relations among the four constants

$$A^{\text{curr}} = \frac{\pi}{4} A^{\text{spin}}, \quad A^{\text{EM}} = 4A^{\text{top}}. \quad (9)$$

These relations reflect the linking between the even and the odd particle members of an isospin family, which results from the clustering relations obeyed by the exact form factors [1]. Since $(A^{\text{curr}})_{\text{PT}} = 1/3\pi$ the first equation in (9) gives $A^{\text{spin}} = \lambda_1$ as in (5), while the second equation fixes the physical normalization of the TC matrix elements in terms of that of the EM tensor, which in our conventions amounts to $\lambda_0 = \frac{1}{4}$.

The λ_0 value can be tested independently by means of Monte Carlo (MC) simulations. In Fig. 2 the results for the two-point functions of the TC density are shown—once computed via the form factor approach, with the absolute normalization fixed according to (5) and once via MC simulations. The simulations were done using the cluster algorithm of [4] with the standard action and the geometrical definition of the TC density. The data in Fig. 2 correspond to a 460 square lattice at inverse coupling $\beta = 1.80$ (correlation length 65.05). The statistical errors are smaller than the size of the dots. The nice agreement confirms $\lambda_0 = \frac{1}{4}$ and hence provides further support for the scaling hypothesis.

The plateau phenomenon for the spectral densities has a counterpart for the two-point functions. In momentum space the latter can be computed as Stieltjes transforms of the corresponding spectral densities. Inserting the decomposition (1) one obtains for the $n \leq N$ particle approximants

$$I_N(y) = \sum_1^N \int_0^\infty d\mu \frac{\rho^{(n)}(\mu)}{\mu^2 + m^2 e^{2y}}, \quad y = \ln p/m. \quad (10)$$

For $n > n_0$ the n -particle spectral densities are evaluated

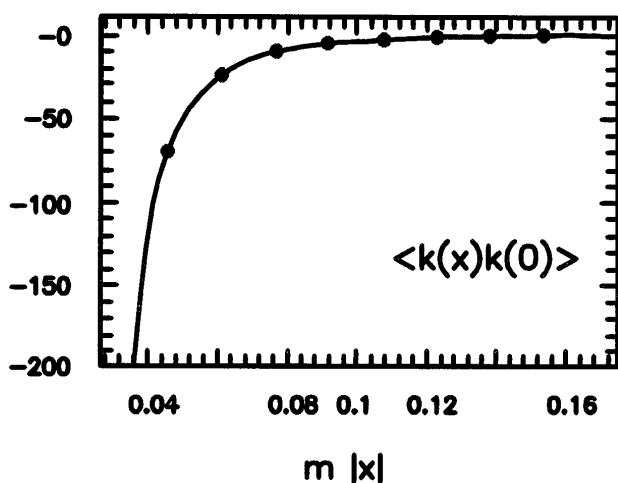


FIG. 2. Two-point function of the topological charge density. Comparison MC data and form factor result. Test of $\lambda_0 = \frac{1}{4}$.

by means of the scaling hypothesis. The $N \rightarrow \infty$ limit of (10) yields a candidate for the exact two-point function.

For the case of the spin field the result is shown in Fig. 3 in units of the two-loop PT result for the same quantity. The “exact” $N = \infty$ curve approaches its asymptotic value (here normalized to unity by using $\lambda_1 = 4/3\pi^2$ for the normalization of the PT result) fairly slowly. Demanding a 1% accuracy for PT one finds that there are two such regimes, one at an intermediate energy scale $4 \lesssim y \lesssim 9$ and a second one for $y \gtrsim 75$. A similar behavior is found for the current two-point function. Thus, somewhat surprisingly, PT does not necessarily improve monotonically with increasing energy. As a reservation one must add, however, that these are 1%–2% effects which may be affected by subleading terms in the scaling law (b). [Inclusion of a three-loop PT contribution does not remove the effect; subleading terms arising from extrapolating $Y^{(n+1)}(z) - Y^{(n)}(z)$ have been taken into account in Fig. 3.]

Relation to KNO scaling.—A bonus of the above scaling hypothesis is that it implies KNO-like scaling laws for multiparticle production processes in nonintegrable perturbations of the model (but not vice versa). To allow for particle production we enlarge our model world by a “weak” sector so that a general state is of the form $|s; w\rangle$, where s is the “strong” (sigma-model) part of the state and w is the weak part. Adding a current-current interaction term, production processes $|s; w\rangle \rightarrow |s'; w'\rangle$ (where the states s and s' have different particle numbers) become possible. The corresponding transition amplitude, to lowest order in the new interaction, reads

$$\mathcal{A} = \int d^2x l^{a\mu}(x) \langle s' | j_\mu^a(x) | s \rangle, \quad (11)$$

where the Fourier transform of $l_\mu^a(x)$ has support at the momentum Q , the weak momentum transfer.

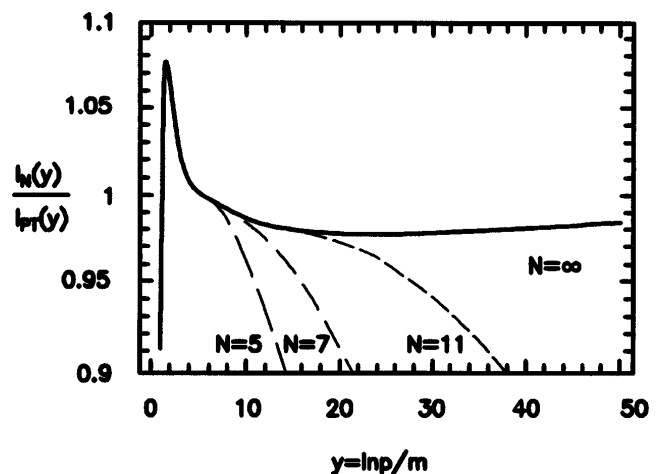


FIG. 3. Spin two-point function: Ratio of $I_N(y)$ approximants and two-loop perturbation theory $I_{\text{PT}}(y)$ versus $y = \ln p/m$.

The simplest production process is the two-dimensional analog of the e^+e^- annihilation. We model this process by choosing $Q^2 > 0$ and $s = |0\rangle$. Summing over all (discrete and continuous) quantum numbers of the n -particle state s' , the probability distribution for producing n particles at energy $\mu = \sqrt{Q^2}$ becomes independent of the weak part of the process and is proportional to the current spectral density. The proportionality factor involves the coupling constant of the external current-current interaction, which drops out when considering the conditional probability P_{2n} for having exactly $2n$ particles produced, once the process has taken place at all. One has

$$P_{2n} = \frac{\mu}{\kappa} \rho_1^{(2n)}(\mu), \quad \sum_{n=1}^{\infty} P_{2n} = 1, \quad (12)$$

where the second equation fixes the normalization constant κ . Similarly, production processes upon perturbation with the EM tensor can be studied, in which case $\rho_0^{(2n)}(\mu)$ appears in (12). Using our scaling hypothesis the energy dependence of the probability distribution can in both cases be expressed, for large μ (where one can approximate the sums by integrals), in terms of $\bar{n} = \sum_{n=1}^{\infty} 2nP_{2n} \sim (\ln \mu)^{1/(1+\alpha)}$, the average number of particles produced. The asymptotic distribution takes the KNO-scaling form [5]

$$\bar{n}P_{2n} = f\left(\frac{n}{\bar{n}}\right). \quad (13)$$

The scaling function $f(q)$ is given in terms of the universal shape function as

$$f(q) = \frac{2(1+\alpha)\tilde{h}_l}{h_l^2} \left(\frac{h_l}{2\tilde{h}_l q}\right)^{\gamma_l} Y_l \left[\left(\frac{h_l}{2\tilde{h}_l q}\right)^{1+\alpha} \right], \quad (14)$$

where the parameters h_l and \tilde{h}_l are the $(\frac{\gamma_l-1}{1+\alpha} - 1)$ th and $(\frac{\gamma_l-2}{1+\alpha} - 1)$ th moments of the universal shape function $Y_l(z)$, respectively, and the exponents are given in (8). The case $l = 1$ corresponds to the current perturbation and $l = 0$ to the perturbation by the EM tensor. These KNO-type scaling laws, however, are valid only for simultaneously large particle numbers and large energies in the perturbed theory. In contrast, the scaling hypothesis (a),(b) for the spectral densities is valid already in the unperturbed theory for all energies, and in particular is nonperturbative in nature.

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*On leave of absence from the Research Institute for Particle and Nuclear Physics, Budapest, Hungary.

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