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Vortex Velocities in the $O(n)$ Symmetric Time-Dependent Ginzburg-Landau Model

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An explicit expression for the vortex velocity field as a function of the order parameter field is derived for the case of point defects in the $O(n)$ symmetric time-dependent Ginzburg-Landau model. This expression is used to find the vortex velocity probability distribution in the Gaussian closure approximation in the case of phase ordering kinetics for a nonconserved order parameter. The velocity scales as L^{-1} in scaling regime where $L \approx t^{1/2}$ and t is the time after the quench. [S0031-9007(96)02240-5]

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The importance of the role of defects in understanding a variety of problems in physics is clear. In certain cosmological [1] and phase ordering [2] problems key questions involve an understanding of the evolution and correlation among defects like vortices, monopoles, disclinations, etc. In studying such objects in field theory questions arise as to how one can define quantities like the density of vortices and an associated vortex velocity field. The purpose of this Letter is to identify the appropriate vortex-velocity field in the context of an $O(n)$ symmetric time-dependent Ginzburg-Landau (TDGL) model for the case of point defects where $n = d$ and d is the spatial dimensionality. Using this rather general definition for the velocity field the distribution of velocities is determined in the case of the late state phase ordering using the Gaussian closure approximation for a nonconserved order parameter. The physical results are that the velocity scales as $L(t)^{-1}$, where $L(t) \approx t^{1/2}$ is the characteristic scaling length for the order parameter correlation function which grows with time t after the quench. The vortex velocity probability distribution function is given in this approximation by

$$P(\vec{v}_0) = \frac{\Gamma(\frac{n}{2} + 1)}{(\pi \overline{v}^2)^{n/2}} \frac{1}{[1 + (\vec{v}_0)^2/\overline{v}^2]^{(n+2)/2}}, \tag{1}$$

where the parameter $\overline{v}$ is defined below and varies as L^{-1} for long times.

The focus here is on the defect dynamics generated by the TDGL model satisfied by a nonconserved n-component vector order parameter $\vec{\psi}(\vec{r}, t)$:

$$\frac{\partial \vec{\psi}}{\partial t} = \vec{K} \equiv -\Gamma \frac{\delta F}{\delta \vec{\psi}} + \vec{\eta}, \tag{2}$$

where Γ is a kinetic coefficient, F is a Ginzburg-Landau effective free energy assumed to be of the form

$$F = \int d^d r \left[\frac{c}{2} (\nabla \vec{\psi})^2 + V(|\vec{\psi}|) \right], \tag{3}$$

where $c > 0$ and the potential is assumed to be of the degenerate double-well form. $\vec{\eta}$ is a thermal noise which is related to Γ by a fluctuation-dissipation theorem.

Consider a system with $n = d$ where there are topologically stable point defects [3] formed in a phase ordering system (quenched, for example, from a high temperature disordered state to a temperature below the order temperature). As pointed out by Halperin [4], and exploited by Liu and Mazenko [5], the vortex density for such a system can be written as

$$\rho = \delta(\vec{\psi}) \mathcal{D}, \tag{4}$$

where $\mathcal{D}$ is the Jacobian (determinant) for the change of variables from the set of vortex positions $r_i(t)$ (where $\vec{\psi}$ vanishes) to the field $\vec{\psi}$:

$$\mathcal{D} = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \nabla_{\mu_1} \psi_{\nu_1} \nabla_{\mu_2} \psi_{\nu_2} \cdots \nabla_{\mu_n} \psi_{\nu_n}, \tag{5}$$

where $\epsilon_{\mu_1,\mu_2,\ldots,\mu_{hn}}$ is the n-dimensional fully antisymmetric tensor and summation over repeated indices here and below is implied.

The first goal here is to derive the equation of motion satisfied by ρ. Toward this end one needs two identities whose proof is relatively straightforward. The first identity is given by

$$\frac{\partial \mathcal{D}}{\partial t} = \nabla_\alpha J_\alpha^{(K)} \qquad \text{Identity I}, \tag{6}$$

where, for a general vector $\vec{A}$, the current $J_\alpha^{(A)}$ is defined as

$$J_\alpha^{(A)} = \frac{1}{(n-1)!} \epsilon_{\alpha,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots\nu_n} A_{\nu_1} \times \nabla_{\mu_2}\psi_{\nu_2} \cdots \nabla_{\mu_n}\psi_{\nu_n}. \tag{7}$$

Identity I is just a statement that the determinant $\mathcal{D}$ is a conserved invariant. Notice that the superscript K on J in this identity is defined by the right hand side of Eq. (2). The second identity takes the from for general vector $\vec{A}$:

$$J_\alpha^{(A)} \nabla_\alpha \psi_\beta = A_\beta \mathcal{D}. \qquad \text{Identity II} \tag{8}$$

Identity II, after using the chain rule for differentiation, leads directly to the result

$$\mathcal{D} \frac{\partial}{\partial t} \delta(\vec{\psi}) = J_\beta^{(K)} \nabla_\beta \delta(\vec{\psi}). \tag{9}$$

When this result is combined with Identity I, one easily obtains the equation of motion for the vortex density

$$\frac{\partial \rho}{\partial t} = \nabla_\beta [\delta(\vec{\psi}) J_\beta^{(K)}]. \tag{10}$$

This continuity equation reflects the fact that the vortex *charge* is conserved. A key point here is that $J_\beta^{(K)}$ is multiplied by the vortex locating δ function. This means that one can replace $\vec{K}$ in $\vec{J}^{(K)}$ by the part of $\vec{K}$ which does not vanish as $\vec{\psi} \to 0$. Thus in the case of a nonconserved order parameter one can replace $J_\beta^{(K)}$ in the continuity equation by

$$J_\beta^{(2)} = \frac{1}{(n-1)!} \epsilon_{\beta,\mu_2,\ldots,\mu_n} \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} [\Gamma c \nabla^2 \psi_{\nu_1} + \eta_{\nu_1}] \times \nabla_{\mu_2}\psi_{\nu_2} \cdots \nabla_{\mu_n}\psi_{\nu_n}. \tag{11}$$

In the case of a conserved order parameter the current for ρ is more complicated because of the overall gradients acting in $\vec{K}$. Because of the standard form of the continuity equation Eq. (10), it is clear that one can identify the vortex velocity field as

$$v_\alpha = -\frac{J_\alpha^{(2)}}{\mathcal{D}}, \tag{12}$$

where it is assumed that the velocity field is used inside expressions multiplied by the vortex locating δ function. This is the primary result of this Letter. It gives one an explicit expression for the vortex velocity field in terms of the original order parameter field. In the particular case of $n = d = 2$ one has the more explicit result (in the absence of noise)

$$v_\alpha = \Gamma c \frac{\epsilon_{\alpha\mu}(\nabla_\mu \psi_x \nabla^2 \psi_y - \nabla_\mu \psi_y \nabla^2 \psi_x)}{\epsilon_{\nu\sigma} \nabla_\nu \psi_x \nabla_\sigma \psi_y}. \tag{13}$$

Notice that the result given by Eq. (10) does not depend on the details of the TDGL model, only that the equation of motion is first order in time.

The expression given by Eq. (12) for the velocity is very useful because it avoids the problem of having to specify the positions of the vortices explicitly. The positions are implicitly determined by the zeros of the order parameter field. The general expression with $J_\alpha^{(K)}$ should be useful in looking at the motion of vortices in the presence of external fields beyond a growth kinetics context.

The practical usefulness of the result Eq. (12) can be seen by asking the question: In the scaling regime of a phase ordering system with point defects, what is the probability of finding a vortex with a velocity $\vec{v}_0$? This probability distribution function is defined by

$$n_0 P(\vec{v}_0) \equiv \langle n \delta(\vec{v}_0 - \vec{v}) \rangle, \tag{14}$$

where $\vec{v}_0$ is a reference velocity, $n = \delta(\vec{\psi}) |\mathcal{D}|$ is the unsigned defect density, $n_0 = \langle n \rangle$, and $\vec{v}$ is given by Eq. (12). We determine P using the Gaussian closure method [6–9] which has been successful in determining the scaling function for the order parameter correlation function. The first step is to express the order parameter in terms of an auxiliary field $\vec{m}$ which is assumed, to a first approximation, to have a Gaussian distribution. In the theory developed in Ref. [7], the relationship between the order parameter and the auxiliary field is given as a solution to the classical interface equation

$$\nabla_m^2 \vec{\psi}(\vec{m}) = V'(|\psi|) \hat{\psi}, \tag{15}$$

where the auxiliary field serves as the coordinate labeling the distance to the defect nearest to space point $\vec{r}$ at time t. The solution of this equation for a charge one vortex is of the form

$$\vec{\psi}(\vec{m}) = A(|\vec{m}|) \hat{m}, \tag{16}$$

where $A(|\vec{m}|)$ vanishes linearly with m for small m with the next term of $\mathcal{O}(m^3)$. It is then easy to show that one can replace $\vec{\psi}$ by $\vec{m}$ in the expression for $\vec{v}$. One can determine $P(\vec{v}_0)$ by first evaluating the more general probability distribution

$$G(\xi, \vec{b}) = \langle \delta(\vec{m}) \delta(\xi_\mu^\nu - \nabla_\mu m_\nu) \delta(\vec{b} - \nabla^2 \vec{m}) \rangle, \tag{17}$$

since

$$n_0 P(\vec{v}_0) = \int d^n b \prod_{\mu,\nu} d\xi_\mu^\nu |\mathcal{D}(\xi)| \delta[\vec{v}_0 - \vec{v}(\vec{b}, \xi)] \times G(\xi, \vec{b}), \tag{18}$$

where

$$\vec{v}(\vec{b},\xi) = -\frac{\vec{J}^{(2)}(\vec{b},\xi)}{\mathcal{D}(\xi)} \tag{19}$$

with

$$\mathcal{D}(\xi) = \frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\xi^{\nu_1}_{\mu_1}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n} \tag{20}$$

and

$$J^{(2)}_{\alpha}(\vec{b},\xi) = \frac{1}{(n-1)!}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \times \Gamma c b_{\nu_1}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n}. \tag{21}$$

In this last expression it has been assumed that the quench is to zero temperature [10] so that the noise can be set to zero. The Gaussian average [11] determining $G(\xi,\vec{b})$ is relatively straightforward to evaluate with the result:

$$G(\xi,\vec{b}) = \frac{1}{(2\pi S_0)^{n/2}}\frac{e^{-\frac{1}{2\overline{S}_4}\vec{b}^2}}{(2\pi\overline{S}_4)^{n/2}}\frac{1}{(2\pi S^{(2)})^{n^2/2}} \times \exp\left[-\frac{1}{2S^{(2)}}\sum_{\mu,\nu}(\xi^{\nu}_{\mu})^2\right], \tag{22}$$

where $S_0 = \frac{1}{n}\langle\vec{m}^2\rangle$ is proportional to L^2,

$$S^{(2)} = \frac{1}{n^2}\langle(\nabla\vec{m})^2\rangle, \tag{23}$$

and

$$\overline{S}_4 = \frac{1}{n}\langle(\nabla^2\vec{m})^2\rangle - \frac{[nS^{(2)}]^2}{S_0}. \tag{24}$$

The quantities S_0, $S^{(2)}$, $\overline{S}_4$ are determined from the theory for the order parameter correlation function. Using this result for $G(\xi,\vec{b})$ in the expression for the probability distribution one can use the usual integral representation for the δ function to perform the integration over the $\vec{b}$ field to obtain

$$n_0P(\vec{v}_0) = \int\prod_{\mu,\nu}d\xi^{\nu}_{\mu}\frac{|\mathcal{D}(\xi)|}{(2\pi S^{(2)})^{n^2/2}}\exp\left[-\frac{1}{2S^{(2)}}\sum_{\mu,\nu}(\xi^{\nu}_{\mu})^2\right]\frac{1}{(4\pi^2S_0\gamma)^{n/2}}\frac{1}{\sqrt{\det M}}\exp\left[-\frac{1}{2\gamma}\sum_{\mu,\nu}v^{\mu}_0[M^{-1}]_{\mu,\nu}v^{\nu}_0\right], \tag{25}$$

where $\gamma = \overline{S}_4(\Gamma c)^2$, and the matrix M is given by

$$M_{\alpha,\beta} = \frac{1}{\mathcal{D}^2[(n-1)!]^2}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu,\nu_2,\ldots,\nu_n}\xi^{\nu_2}_{\mu_2}\cdots\xi^{\nu_n}_{\mu_n} \times \epsilon_{\beta,\mu'_2,\ldots,\mu'_n}\epsilon_{\nu,\nu'_2,\ldots,\nu'_n}\xi^{\nu'_2}_{\mu'_2}\cdots\xi^{\nu'_n}_{\mu'_n}. \tag{26}$$

It is straightforward to obtain the rather clean results

$$\det(M) = \frac{1}{(\mathcal{D})^2} \tag{27}$$

and

$$M^{-1}_{\alpha\beta} = \sum_{\nu}\xi^{\nu}_{\alpha}\xi^{\nu}_{\beta} \tag{28}$$

so that

$$n_0P(\vec{v}_0) = \int\prod_{\mu,\nu}d\xi^{\nu}_{\mu}\frac{1}{(2\pi S^{(2)})^{n^2/2}} \times \exp\left[-\frac{1}{2S^{(2)}}\sum_{\mu,\nu}(\xi^{\nu}_{\mu})^2\right]\frac{\mathcal{D}^2}{(4\pi^2S_0\gamma)^{n/2}} \times \exp\left[-\frac{1}{2\gamma}\sum_{\alpha,\beta,\nu}v^{\alpha}_0\xi^{\nu}_{\alpha}\xi^{\nu}_{\beta}v^{\beta}_0\right]. \tag{29}$$

The remaining integrals can be reduced to a separable product of Gaussian integrals through a linear transformation with the result

$$n_0P(\vec{v}_0) = \left(\frac{S^{(2)}}{2\pi S_0}\right)^{n/2}\left(\frac{1}{2\pi\overline{v}^2}\right)^{n/2} \times \frac{n!}{[1+(\vec{v}_0)^2/\overline{v}^2]^{(n+2)/2}}, \tag{30}$$

where

$$\overline{v}^2 = \gamma/S^{(2)} = (\Gamma c)^2\frac{\overline{S}_4}{S^{(2)}}. \tag{31}$$

Since $P(v_0)$ is normalized to one, we find on integration over $\vec{v}_0$ the result

$$n_0 = \left(\frac{S^{(2)}}{2\pi S_0}\right)^{n/2}\frac{n!}{2^{n/2}\Gamma(\frac{n}{2}+1)}, \tag{32}$$

which agrees with the result found by Liu and Mazenko [5] using a more indirect method. After using this result for n_0 one finally obtains the result given by Eq. (1). This result basically says that the probability of finding a large velocity decreases with time. However, since this distribution falls off only as $v_0^{-(n+2)}$ for large v_0 only the first moment beyond the normalization integral exists. Bray [12] has a scaling argument associated with vortex-antivortex final annihilation which leads to this large velocity tail.

The determination of S_0, $S^{(2)}$, $\overline{S}_4$, and $\overline{v}$ requires a theory for the auxiliary field correlation function

$$C_0(12) = \frac{1}{n}\langle\vec{m}(1)\cdot\vec{m}(2)\rangle. \tag{33}$$

There are two theories available and both are of the Gaussian closure type assumed above. One, due to Ohta, Jasnow, and Kawasaki (OJK) [13], essentially postulates that $C_0(12)$ is a Gaussian

$$C_0(12) = S_0 e^{-\vec{r}^2/2L^2}, \tag{34}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$. In this case one easily finds that

$$\overline{v}^2 = \frac{2d}{L^2}(\Gamma c)^2, \tag{35}$$

where the coefficient of L^2/t is undetermined in the theory of OJK. Using the theory developed in Ref. [14] for $n = 2$ one finds self-consistently [15] that

$$\overline{v}^2 = \left(1 + \frac{\pi}{4\mu}\right)\frac{(\Gamma c)^2}{t}, \tag{36}$$

where $\mu = 0.53721\ldots$ is the eigenvalue determined within the theory [14].

One can go forward and extend these ideas to treat two-point velocity correlation functions and stringlike defects ($n = d - 1$) as will be discussed elsewhere.

I thank Professor Alan Bray for useful comments on this work.

[1] For a recent review of the role of defects in both a cosmological and condensed matter context, see *Formation and Interactions of Topological Defects,* edited by A.-C. Davis and R. Brandenberger, NATO ASI, Ser. B, Vol. 349 (Plenum, New York, 1995).
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[10] Because of the growing length L in the problem the role of temperature is typically irrelevant as long as the final temperature T is less than the critical temperature T_c.
[11] In general the average is over noise and initial conditions. When the noise is set to zero then the only disordering agents are the initial conditions. The random initial conditions act just like noise localized at some time. However, once the ordering has grown sufficiently large it has lost memory of the specific nature of the initial conditions. This was investigated in detail in Ref. [6]. It is known from the work of C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978) that in such problems one can change from averages over the noise and initial conditions to averages over the fields.
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[15] The result $\overline{S}_4 \approx L^{-2}$ is nontrivial and assumes that there is no correction to scaling terms of the form $C_0 = S_0 - \frac{S^{(2)}}{2}(r^2 + b_4 r^4 + \ldots)$ for small r where b_4 is a constant as $t \rightarrow \infty$. A detailed self-consistent analysis shows that $b_4 = 0$ if $\overline{S}_4$ is to be positive at late times.