Scaling and Nucleation in Models of Earthquake Faults

W. Klein,[1] J. B. Rundle,[2] and C. D. Ferguson[1]
[1]*Physics Department, Center for Polymer Physics and Center for Computational Science, Boston University, 590 Commonwealth Avenue, Boston, Massachusetts 02215*
[2]*Department of Geological Sciences, Physics Department and CIRES, University of Colorado, Boulder, Colorado 80309*
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We present an analysis of a slider block model of an earthquake fault which indicates the presence of metastable states ending in spinodals. We identify four parameters whose values determine the size and statistical distribution of the "earthquake" events. For values of these parameters consistent with real faults we obtain scaling of events associated not with critical point fluctuations but with the presence of nucleation events. [S0031-9007(97)03140-2]

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Since Gutenberg and Richter [1] (GR) noticed that the energy released in earthquakes obeys a scaling law researchers have tried to explain the origin of that scaling. To study the statistical aspects of earthquake events Burridge and Knopoff [2] (BK) proposed a slider block model amenable to numerical and analytic investigation that hopefully contained the essential physics of faults. The dynamics of this model has been the subject of considerable interest, initially among seismologists [3–6] and more recently in the condensed matter community [7–10]. Although cluster scaling is obtainable from the BK model there has been no clear connection between the scaling and any underlying critical phenomena. In addition the clusters observed in the BK model exhibit a compact structure rather than the fractal morphology associated with critical fluctuations. Moreover, earthquake phenomena are considerably more complicated than scaling plots alone indicate. In addition to scaling there are faults that show only quasicontinuous motion (creep) and evidence that at least some earthquakes are nucleation events [11]. In order to gain deeper understanding of the range of phenomena obtainable from the BK model we have investigated a cellular automaton version introduced by Rundle *et al.* [5,6] (RJB) with the modification that we concentrate on systems with long range interactions. Our main results are as follows: (a) The model has two critical points at which the cluster scaling is associated with "thermal" critical phenomena. (b) There is, under the proper conditions, a metastable, ordered, high stress state and a stable, disordered, lower stress state, both of which can be described by an equilibrium theory [12]. (c) Nucleation events occur which are similar to those discussed in Ref. [11].

The RJB model also has a two dimensional array of blocks connected by springs with spring constants K_C. Each block is also connected to a loader plate, which moves at a velocity V, by a spring with constant K_L. The system is initialized by random positioning of the blocks and the plate is moved a distance $V\Delta T$, where $\Delta t = 1$. The stress σ_i of each block is checked to ascertain if it exceeds a prescribed failure threshold σ_i^F. If so it is moved a distance prescribed by a "jump rule" specified below. The process of moving blocks if their stress exceeds σ_i^F continues until all blocks have $\sigma_i < \sigma_i^F$. Then the plate is again moved a distance $V\Delta t$ and the entire process repeats. Measurements are generally made of clusters of failed sites where a cluster is defined as failed sites connected by springs to each other. The motion of the jth block is defined by the equations

$$U_j(t+1) = U_j(t) + \left[\frac{\sigma_j(t) - \sigma_j^R}{K}\right]\Theta(\sigma_j(t) - \sigma_j^F) + \eta_j(t) \tag{1}$$

and

$$\sigma_i(t) = \sum_j T_{ij} U_j(t) + K_L V \sum_n \Theta(n - t). \tag{2}$$

Here $U_j(t)$ is the position of the jth block at time t, σ_j^R is the residual stress to which the block is set after it fails (σ_j^R and σ_j^F will be taken to be constants in this work) $K = K_L + \sum_{j,i\neq j} T_{ij}$ and $\Theta(x) = 0$ if $x \leq 0$ and is equal to one if $x > 0$. The stress Green function T_{ij} is taken to be that of a linear elastic medium [13], i.e., $T_{ij} \sim 1/|i-j|^3$ with both an infrared and ultraviolet cutoff and $\eta_i(t)$ is a random noise with an amplitude set by a parameter β.

Multiplying both sides of Eq. (1) by T_{ij}, summing over j, and using Eq. (2) we have

$$\sigma_i(t+1) - \sigma_i(t) = \frac{1}{K}\sum_j T_{ij}[\sigma_j(t) - \sigma^R] \times \Theta(\sigma_j(t) - \sigma^F) + K_L V + \eta_i'(t), \tag{3}$$

where $\eta_i'(t) = \sum_j T_{ij}\eta_j(t)$. With $\eta_j(t) = 0$, nearest neighbor springs ($q = 4$), and $V \sim 0$ this model gives the same qualitative results, with respect to the statistical distribution of "earthquakes" as the BK model which is also a $q = 4$ model in two dimensions [5,6,10].

Deviating slightly from the original BK idea we take $q \gg 1$, which is dictated by the long range nature of T_{ij}. That is, we adopt a model with each block directly connected to many other blocks via springs with constants K_C. The importance of $q \gg 1$ that the $q \to \infty$ limit generates mean-field (MF) behavior [14,15] which can be qualitatively different [16,17] than that of the $q = 4$ model. In particular, critical exponents will differ.

We are interested only in phenomena on length scales larger than the range of interaction $q^{1/2}$. With this in mind we develop a coarse-grained [18] description of the $q \gg 1$ BK-RJB model that retains the physics on these length scales. To do this we average over the blocks within a course-grained cell centered at i, taking the volume of the cell to be q, and over a course-grained time τ. The left hand side of Eq. (3) retains the same form, but the variable $\sigma_i(t)$ is replaced by $\overline{\sigma}_i(\tau)$. Additional considerations arise in coarse graining the right hand side. First we expand the Fourier transform of T_{ij} (assumed to be analytic due to the cutoffs) in a power series in the transform variable $|\vec{k}| = k$ and truncate the series at k^2. Inverting the Fourier transform we obtain

$$\sum_j T_{ij}\sigma_j(t)\Theta\big(\sigma_j(t) - \sigma_j^F\big) \sim -qK_C \sum_j \Delta_{ij}\sigma_j(t)\Theta\big(\sigma_j(t) - \sigma^F\big) - K_L\sigma_i(t)\Theta\big(\sigma_i(t) - \sigma^F\big), \tag{4}$$

where $-K_L$ and qK_C are the zeroth and second moments of T_{ij}, respectively, q is the number of blocks that a single block is connected to by springs, and Δ_{ij} is the matrix (discrete) representation of the Laplacian. Note that the sum preceding the Laplacian is not over nearest neighbors, but over coarse-grained blocks whose length scale is set by $q^{1/2}$.

The second step in the coarse-graining procedure is to do a partial sum in Eq. (3) over those blocks in a volume of size q, centered at block i, that fail in a coarse-grained time interval. To perform this step we note that in the MF ($q \to \infty$) limit the time averaged stress on a block from the connector springs (with spring constant K_C) will become extremely small. This is expected from symmetry but to confirm this point we measured the mean stress on the blocks and compared it to K_L times the mean distance between the actual position of the blocks and the position at which the force from the loader spring would be zero. These values approach each other within a coarse-graining time as q increases [19] indicating that the loader plate springs account for almost all the stress on a block as $q \to \infty$. Standard MF arguments [20] indicate that the fluctuations in the stress from the mean value go to zero as $q^{-1/2}$.

Since the blocks are weakly interacting in the MF limit, we expect, from the central limit theorem, that within the coarse-grained volume of size q, on a time scale short compared to the coarse-grained time, the stress distribution of the blocks will equilibrate to a Gaussian centered about $\overline{\sigma}(\vec{x},\tau)$, where $\vec{x}$ labels the coarse-grained volume and τ is the coarse-grained time scale. The partial sum in Eq. (3) can now be done using the Gaussian distribution; i.e.,

$$\frac{1}{q}\sum_j{}'[\sigma_j(t) - \sigma^R]\Theta\big(\sigma_j(t) - \sigma_j^F\big) \sim (\sigma^F - \sigma^R)\frac{\sqrt{\beta}}{\sqrt{\pi}}\int_{\sigma^R}^{\sigma^F} d\sigma\, \exp\{-\beta[\sigma - \overline{\sigma}(\vec{x},\tau)]^2\}, \tag{5}$$

where the prime on the sum denotes the sum over the blocks that fail inside the coarse-grained volume in a coarse-grained time unit. The parameter σ_0, which specifies the number of failed blocks in a coarse-grained time, remains to be determined. We have assumed that blocks fail at most once after a plate update, which can be shown [19] to be true in the $q \to \infty$ limit for $V < (\sigma^F - \sigma^R)/K$. The factor $\sqrt{\beta}/\sqrt{\pi}$ is an approximation to the normalization for $\beta \gg 1$, the range of β we will treat in this Letter. Large [21] β results in a narrow Gaussian (see Fig. 1) so that replacing σ_F by infinity causes negligible error in the normalization.

We define $N\big(\overline{\sigma}(\vec{x},\tau)\big)$ to be the number of ways the stress $q\overline{\sigma}(\vec{x},\tau)$ can be distributed among the q blocks in the coarse-grained volume. Using the idea [22] that the log of a probability distribution is proportional to the potential of a generalized force we include a term [19]

$$\begin{aligned}\frac{\delta S(\overline{\sigma}(\vec{x},\tau))}{\delta\overline{\sigma}(\vec{x},\tau)} = -\frac{\delta \ln N\big(\overline{\sigma}(\vec{x},\tau)\big)}{\delta\overline{\sigma}(\vec{x},\tau)} &= \frac{\beta^{-1}}{\sigma^F - \sigma^R}\ln\left[\frac{\overline{\sigma}(\vec{x},\tau) - \sigma^R}{\sigma_F - \overline{\sigma}(\vec{x},\tau)}\right] \\ &- \frac{\beta^{-1}}{\sigma^F - \sigma^R}\left(\frac{\beta}{\pi}\right)^{1/2}\int_{\sigma^R}^{\sigma^F} d\sigma\, \ln\left(\frac{\sigma - \sigma^R}{\sigma_F - \sigma}\right)\exp\{-\beta[\sigma - \overline{\sigma}(\vec{x},\tau)]^2\}\end{aligned} \tag{6}$$

to the coarse-grained force. The addition of this entropy term follows from the standard coarse-graining assumption that the system equilibrates within the coarse-grained volume on a time scale short compared to the time scales of interest [23]. Equation (6) follows from the assumption of local equilibrium [19].

Combining Eqs. (3)–(6) and taking the continuum limit in time and space

$$\frac{\partial\sigma(\vec{x},\tau)}{\partial\tau} = K_L V + \overline{\eta}(\vec{x},\tau) + \frac{(qK_C\nabla^2 - K_L)}{K}\frac{(\sigma^F - \sigma^R)}{2}(\mathrm{erf}\{-\sqrt{\beta}[\sigma^F - \overline{\sigma}(\vec{x},\tau)]\} - \mathrm{erf}\{\sqrt{\beta}[\sigma_0 - \overline{\sigma}(\vec{x},\tau)]\}) - \frac{\beta^{-1}}{\sigma^F - \sigma^R}\left[\ln\left(\frac{\overline{\sigma}(\vec{x},\tau) - \sigma^R}{\sigma_F - \overline{\sigma}(\vec{x},\tau)}\right) - \left(\frac{\beta}{(\pi)}\right)^{1/2}\int_{\sigma^R}^{\sigma_F} d\sigma\,\ln\left(\frac{\sigma - \sigma^R}{\sigma_F - \sigma}\right)\exp\{-\beta[\sigma - \overline{\sigma}(\vec{x},\tau)]^2\}\right], \quad (7)$$

where $\mathrm{erf}(z)$ is the error function and $\overline{\eta}(\vec{x},\tau)$ is the coarse-grained noise. This is the coarse-grained equation for the RJB model. A detailed examination of its solutions will be presented in Ref. [19]. Here we will examine the properties of the time independent spatially homogeneous solutions. Setting the noise and derivatives to zero, Eq. (7) becomes

$$\frac{K_L}{K}\frac{(\sigma^F - \sigma^R)}{2}\{\mathrm{erf}[\sqrt{\beta}(\sigma^F - \overline{\sigma})] - \mathrm{erf}[\sqrt{\beta}(\sigma_0 - \overline{\sigma})]\} + \frac{\beta^{-1}}{\sigma^F - \sigma^R}\left[\ln\left(\frac{\overline{\sigma} - \sigma^R}{\sigma_F - \overline{\sigma}}\right) - \left(\frac{\beta}{\pi}\right)^{1/2}\int_{\sigma^R}^{\sigma^F} d\sigma\,\ln\left(\frac{\sigma - \sigma^R}{\sigma_F - \sigma}\right)\exp[-\beta(\sigma - \overline{\sigma})^2]\right] = K_L V\,. \quad (8)$$

To determine σ_0 we note that the blocks can be treated as noninteracting *within the interaction range* for $q \to \infty$. This follows from the observation that if every block interacts with every other block then there is no spatial scale for fluctuations and the interactions can be absorbed into an effective or mean field. For the purposes of calculating MF "thermodynamics" we can assume that every block interacts with all the others in the system [24]. Since the solution of Eq. (8) is the space and time average of $\overline{\sigma}(\vec{x},\tau)$ we must have $\overline{\sigma} = (\sigma^F + \sigma^R)/2$ (see Fig. 1). With this replacement, Eq. (8) can be solved for σ_0. For $\overline{\sigma} = (\sigma^F + \sigma^R)/2$ the entropy term in Eq. (6) equals zero and σ_0 is the solution of

$$\frac{(\sigma^F - \sigma^R)}{2K}\left\{\mathrm{erf}\left(\sqrt{\beta}\,\frac{\sigma^F - \sigma^R}{2}\right) - \mathrm{erf}\left[\sqrt{\beta}\left(\sigma_0 - \frac{\sigma_F + \sigma^R}{2}\right)\right]\right\} = V\,. \quad (9)$$

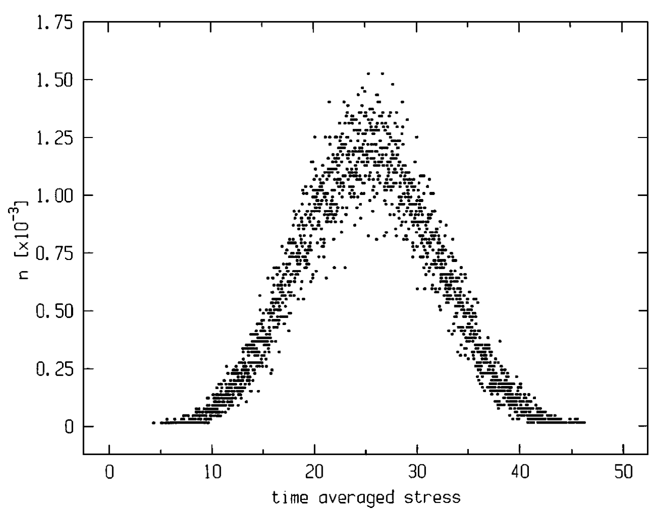

FIG. 1. The number of blocks with time averaged stress $\overline{\sigma}$ per block for the RJB model. The system was run with $V \sim 0$, $\sigma^F = 50$, $\sigma^R = 0$, $K = 9.95$, $K_L = 1$, and random initial conditions. The data were collected over 10 000 plate updates in a system with 256×256 blocks during which 84% of the blocks failed.

For $\beta \gg 1$ and $\sigma^R = 0$, $\mathrm{erf}[\sqrt{\beta}\,\sigma^F/2] \sim 1$ and $\mathrm{erf}[\sqrt{\beta}(\sigma_0 - \sigma^F/2)]$ is approximately a step function equal to minus one for $\sigma_0 < \sigma^F/2$ and one for $\sigma_0 > \sigma^F/2$. Therefore, $\sigma_0 \sim \sigma^F/2$ for $\sigma^F/K > V > 0$. For $V > V_c = \sigma^F/K$, the upper bound of a single failure per block per plate update is no longer valid. As will be discussed in detail in Ref. [19] large events with multiple failures per block occur for $V > V_c$. Solutions for other values of β and σ^R will also be discussed in Ref. [19]. However, small values of the noise are implicit in MF theories [20], where noise amplitudes are scaled by $q^{-1/2}$.

Clearly with this value of $\sigma_0, \overline{\sigma} = \sigma_F/2$ is a solution to Eq. (8). Moreover, if we consider Eq. (8), with $K_L V$ brought to the left hand side, as the derivative of a potential $\Phi(\overline{\sigma})$, it is straightforward to show that $\overline{\sigma} = \sigma_F/2$ is a minimum of $\Phi(\overline{\sigma})$ consistent with Ref. [12]. There is, however, another solution of Eq. (8) with $\sigma_0 = \sigma^F/2$. In Fig. 2 we plot (schematically) the left hand side of Eq. (8) with $\sigma^R = 0$ and $\sigma_0 = \sigma^F/2$. The horizontal straight line is VK_L. The solutions of Eq. (8) are the intersections of the two curves. As can be seen from Fig. 2 there is a high stress low entropy solution in addition to the one at $\overline{\sigma} = \sigma^F/2$ for $V_c > V > V_c/2$.

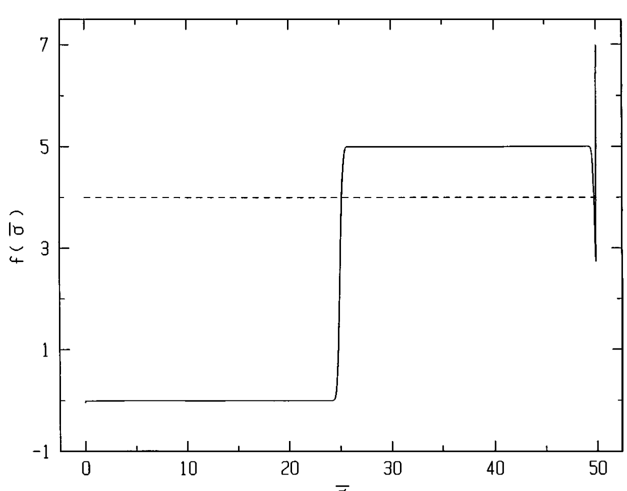

FIG. 2. Schematic plot of $f(\overline{\sigma})$, the left hand side of Eq. (8), for $\beta = 10$.

It can also be shown that this solution is a minimum of $\Phi(\overline{\sigma})$. In addition, the low stress solution is the global minimum for $V < V_c/2 + S(\overline{\sigma})/\sigma^F$, whereas the high stress solution is the global minimum for larger velocities. The high stress solution corresponds to periodic events in which a significant fraction of the blocks fail once after a plate update. The stress is replaced in a sequence of plate updates in which they are few failed blocks.

The cluster scaling exponent, $(n_c(s) \sim s^{-\tau}\tau = 3/2)$ can be obtained from the theory by considering the frequency of nucleation. Since $q \gg 1$ spinodal nucleation is the dominant event [17,24]. The nucleation rate n_c is given by [25,26]

$$n_c \sim \frac{[q(\Delta V)^{3/2-d/4}]^{d/2}}{q(\Delta V)^{-d/4}} \exp[-q\beta(\Delta V)^{3/2-d/4}]. \quad (10)$$

Here $\Delta V = V - V_c$. To convert this function of ΔV to a function of cluster size S requires hyperscaling which is imposed by using q as an auxiliary scaling field to keep $q(\Delta V)^{3/2-d/4}$ fixed [24]. Using Eq. (10) and the fact that the size S of the critical droplet in spinodal nucleation scales [25] as $S \sim q(\Delta V)^{1/2-d/4}$ we obtain, independent of dimension, $n_c \sim S^{-3/2}$ in agreement with our data [19]. The fact that the GR scaling comes from the nucleation droplets rather than the assumption of critical phenomena explains why the earthquakes have a compact rather than a fractal morphology.

In conclusion, we have presented the first explicit coarse-grained theoretical analysis of slider block models incorporating the long range stresses indicated by elasticity theory. The phenomenology of these long range models is often quite different from that of their short range counterparts [12,16,17,20,27]. This analysis has resulted in the identification of scaling with nucleation near the MF spinodal rather than with critical point fluctuations. In addition this analysis forms a bridge between slider blocks models and somewhat more phenomenological continuum models of earthquake faults [28]. It is also interesting to note that large events which are different in character from small events were found in a numerical analysis of the BK model [29]. Our analysis provides a prediction for the scaling exponent of these events which has been confirmed in the RJB model. A similar statistical analysis of the large event statistics in the BK model would prove most interesting.

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