Interacting Pulses in Three-Component Reaction-Diffusion Systems on Two-Dimensional Domains

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(Received 21 November 1996; revised manuscript received 5 March 1997)

We present a three-component reaction-diffusion system capable to support an arbitrary number of interacting traveling pulses in two spatial dimensions. Whereas a global coupling added to a two-component system is able to stabilize a single pulse, a fast and strongly diffusive third component can be used to stabilize multipulse solutions. We study two-pulse scattering including extinction and present a pulse generation process leading to a coherently propagating array. [S0031-9007(97)03097-4]

PACS numbers: 82.20.Mj, 47.35.+i, 82.20.Wt

Nonlinear reaction-diffusion (RD) systems are well suited to model a wide range of physical [1–3], chemical [4,5], and biological [6–8] pattern formation processes. In particular, stationary localized structures (single and multispot patterns) have been obtained in one- and two-dimensional systems in accordance with numerous experimental results. Recent observations in an ac gas discharge between two glass plates [1] established the long-time existence of an almost arbitrary number of moving spots limited only by the size of the system. Repulsion, annihilation, and generation of these spots has been observed. Since moving localized solutions which remain stable could not be obtained in the framework of two-component activator-inhibitor models, we propose a set of three RD equations capable to describe these phenomena at least in a qualitative manner. Our model system is simple enough to motivate the design of further experimental setups for the study of traveling spots in two dimensions. In order to motivate the construction of a three-component model for the description of traveling spots, we start with a short review of the localized structures that have been found using one- and two-dimensional RD models with one and two components, respectively.

Simple front propagation is well described by one-component RD systems [9–12]. The interaction between two such fronts is attractive [13], destabilizing any multifront solution, as well as closed front lines in two dimensions. This restriction can be overcome by introducing a global inhibitory feedback which has proved to be able to change the character of front-front interactions to a repulsive type and thus to stabilize a single localized pattern. Nevertheless, multispot solutions are still unstable in such a model since an antisymmetrical evolution of two localized spots is not affected by a global term, and, therefore, cannot be suppressed. Instead, one of the spots grows, while the other one shrinks and vanishes [14]. It is well known that for one-dimensional and stationary two-dimensional problems the remedy lies in a distributed second component with inhibiting dynamics. In the limit of strong inhibitor diffusion front interaction is repulsive [13], permitting stable stationary multispot solutions on finite domains. There is also an intermediate range of inhibitor diffusion where the type of interaction oscillates between attractive and repulsive according to the distance between neighboring fronts [13] permitting more complicated patterns and, in addition, multifront structures on infinite domains, closely related to the well-studied Turing patterns [15].

Concerning time dependent behavior, pulses in one spatial dimension as well as stripes, spirals, or scroll waves in higher-dimensional spaces have been treated successfully in the frame of two-component approaches [4,16,17]. There is, however, still a lack of suitable models concerning fully localized pulses in two- or higher-dimensional systems. Earlier attempts to treat these traveling spots have been made by various groups on the basis of two-component RD systems with additional global feedback [18–20]. Whereas single pulses are easily produced, it seems that two-pulse solutions, analogous to the case of one-component systems, are always unstable as far as numerical results suggest. In this Letter we propose a three-component RD system to remove this deficiency.

To understand our approach it is essential to have in mind the nature of the stability problem arising in the two-pulse case. To this end we start with an (unstable) single pulse in the following two-component system:

$$u_t = D_u(u_{xx} + u_{yy}) + f(u) - v + \kappa_1\,, \quad (1a)$$

$$\tau v_t = D_v(v_{xx} + v_{yy}) + u - v\,. \quad (1b)$$

In these equations u and v are scalar variables on a two-dimensional domain $\Omega = [0, L] \times [0, L]$ with periodic boundary conditions. Indices t, x, and y denote derivatives with respect to time and space coordinates, and $f(u)$ is a cubic-like function. The nonlinearity is chosen to be $f(u) = \lambda u - u^3$, and we use $\tau \gg 1$. For numerical simulations we used the Crank-Nicolson scheme on a grid with spatial discretization Δx and time step Δt. Figure 1 shows the typical evolution of such an unstable solution moving from the left to the right. The shape of the activator is outlined by the $u = 0$-iso-line, while the gray-scale image shows the distribution of the inhibitor v,

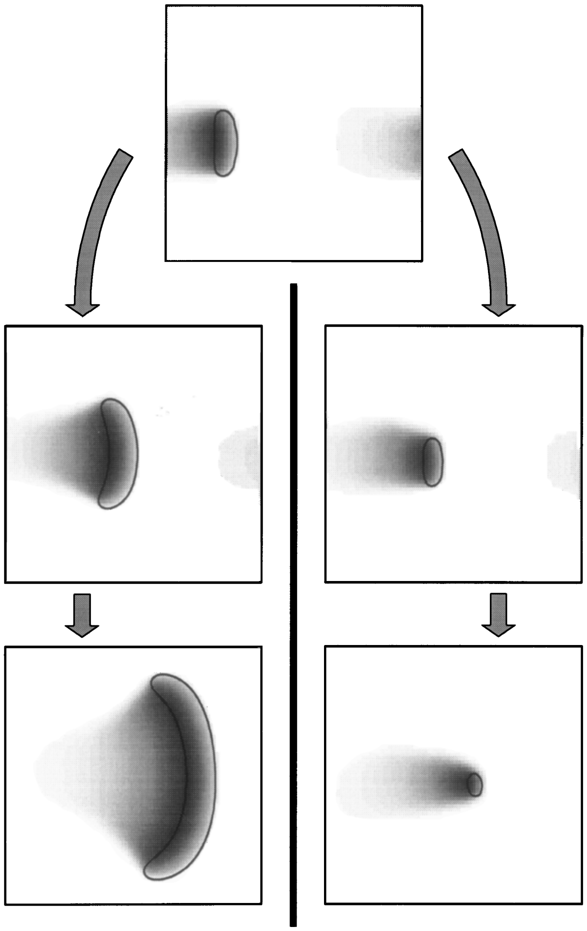

FIG. 1. The typical evolution of a pulse in a two-component RD system in the absence of a global inhibitory coupling usually follows one of the two pathways to the left or to the right. Parameters: $D_u = 10^{-3}$, $D_v = 1.25 \times 10^{-3}$, $\kappa_1 = -0.775$, $\tau = 25$, $\lambda = 2$, $L = 2.3$, $\Delta x = 0.026$, and $\Delta t = 0.035$.

with high concentrations indicated by dark grey. Either the spot shrinks and is extinguished (right) or it grows to a banana shape and tends to form a spiral (left); on a small periodic boundary domain a traveling stripe may also result. Of course, it is possible to have more unstable modes, but this situation seems to be closest to the stable pulse we are heading for. To control the evolution of the unstable mode, it has been suggested to introduce a global feedback, which controls the total amount of the species u in the system. Such a feedback is well known from various experimental setups [21] and can easily be introduced by replacing κ_1 in Eq. (1a) by

$$\kappa_1 = \kappa_1^{\mathrm{new}} - \kappa_2 \frac{1}{|\Omega|} \int_\Omega u\, d\omega \,. \tag{2}$$

Starting with an unstable pulse from system (1) the parameter κ_2 is increased. If κ_1^{new} is adjusted together with κ_2 such that κ_1 remains fixed, the pulse is still a solution of the equations, but its stability properties can change. Simulations show that for a sufficiently large feedback the instability depicted in Fig. 1 is suppressed. Thus a stable moving spot is generated, if no other instability is present.

The instability of the pulse, however, shows up again as soon as there is more than one pulse in the system. The reason is that an antisymmetrical combination of the localized single-pulse modes is not recognized by the integral feedback. Hence, this combination destabilizes the two-pulse solution, at least if the pulses are separated far enough. Numerical simulations with two pulses show that one pulse vanishes, while the other one grows. After the first pulse has disappeared, the other one often remains stable, but is about two times as big as at the beginning. This is due to the global feedback which approximately acts as to keep the excited area of high activator concentration constant. Since this kind of destabilization usually is rather slow, it is possible to examine interactions between these structures before they collapse. Depending on the time scale τ of the inhibitor and on its diffusion-length, head-on collisions that resulted in 180° or 90° scattering have been observed [18,20]. 180° and 90° scattering of two spots was also observed in a two-component model for the evolution of current density distributions in a p-n-p-n diode [22]. In this case Neumann boundary conditions impose a symmetry on the system in such a way that antiphase modes, which can lead to the extinction of one spot, are suppressed. Thus an arbitrary number of consecutive collisions can take place.

Our approach to solve the stability problem is to replace the global coupling by a second inhibitor, which has to be fast and strongly diffusive. Thus we arrive at the three-component system

$$u_t = D_u(u_{xx} + u_{yy}) + f(u) - v - \kappa_3 w + \kappa_1 \,, \tag{3a}$$

$$\tau v_t = D_v(v_{xx} + v_{yy}) + u - v \,, \tag{3b}$$

$$\theta w_t = D_w(w_{xx} + w_{yy}) + u - w \,. \tag{3c}$$

It is simple to prove that in the limit $\theta \to 0$ and $D_w \to \infty$ the third component reproduces the global feedback. Systems of this type are capable of supporting localized moving structures. Figure 2 visualizes the distributions u, v, and w of a pulse after moving through the domain from the left to the right many times. In (a) and (b) the activator is plotted as an iso-line $u(x,y) = 0$, and the fields v and w are displayed as gray-scale images, respectively. The different behavior of the two inhibitors can easily be described: Because of its slow time scale, τ, the first inhibitor is located behind the activator, thus reflecting the direction of motion. This can also be observed in (c), where a cut for $y = 0.5$ is plotted. The second inhibitor surrounds the activator, because it is fast and strongly diffusing. Its task is to inhibit a further extension of the pulse perpendicular to the direction of

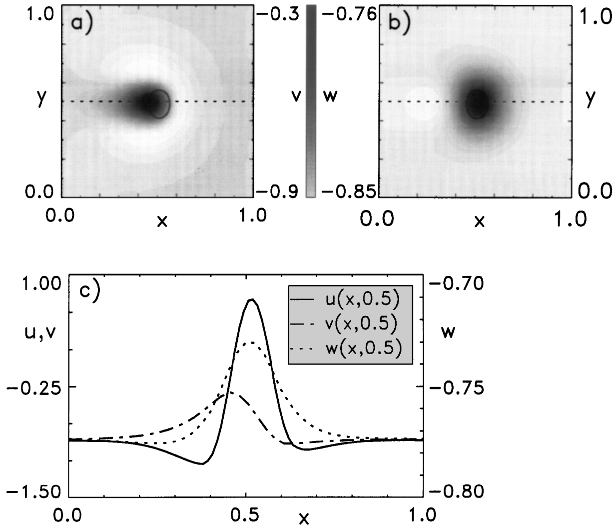

FIG. 2. Typical shape of a traveling three-component spot. Compared to the $u = 0$-iso-line, the distributions of the two inhibitors are shown. (a) The slow component v forms the long tail expected for propagating pulses. (b) The fast component w almost reproduces the shape of the activator distribution, smeared out, however, and strongly flattened as can be seen on the central cut (c) corresponding to the dashed lines in (a) and (b). Parameters are as in Fig. 1, except for $\kappa_1 = -6.92$, $\kappa_3 = 8.5$, $\tau = 48$, $D_w = 0.064$, $\theta = 1$, and $L = 1$.

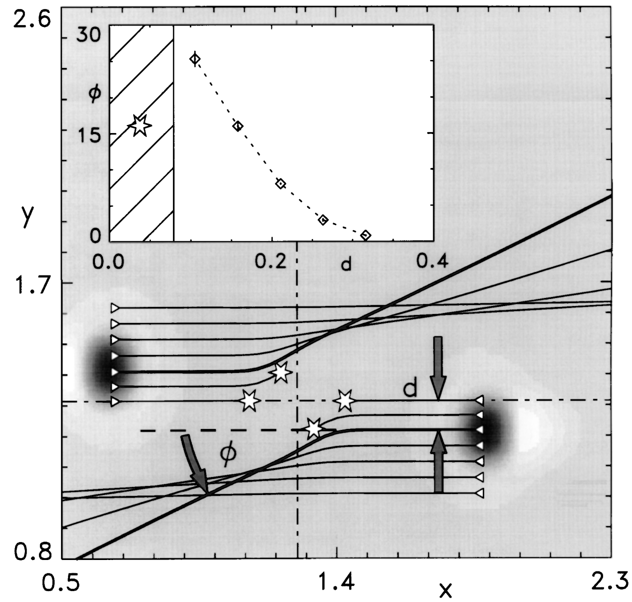

FIG. 3. Collision of two traveling spots. The spots approach each other with antiparallel initial velocities. Depending on the offset d, either the direction of propagation is changed or the pulses annihilate each other. The positions of pulse-annihilation are marked by stars. Scattering angles are presented in the inset as they depend on d. Parameters are as in Fig. 2 except for $L = 2.6$, $\Delta x = 0.018$, and $\Delta t = 0.014$.

motion. We tried to make D_w as small as possible to reduce numerical effort. It turned out that for given values D_u, D_v (Fig. 1), $D_w \approx 0.064$ is near to the lower bound for the existence of traveling pulses. This explains the nearly circular shape of the activator distribution. For $D_w \gg 0.064$ the distribution gets more similar to that of a typical stable spot in the two-component system with global coupling.

Since the equations are local, it is obvious that an arbitrary number of pulses can exist if the system is large enough. With these objects we have carried out simulations of scattering processes. Some of the results are summarized in Fig. 3. In this case we changed the collision parameter d, which describes the shift of the spots perpendicular to their velocity, from zero to 0.33. Choosing $d = 0$ corresponds to a head-on collision. The gray-scale image is the distribution of u for $t = 0$ and $d = 0.11$. In dependence of d two different cases can be found: For $d < 0.08 \pm 0.02$ the two pulses are annihilated, whereas for larger d repulsion leads to a deflection of the spots by an angle $\Phi(d)$. The function $\Phi(d)$ is sketched in the inset of Fig. 3. The star marks the region of pulse extinction. In the remaining region the w distributions of the two objects overlap and lead to an increase of the second inhibitor in front of the spots. Thus the propagation of the activator toward the opposite spot is slowed down and the pulses are deflected.

Annihilation of pulses is observed if the repulsion due to the inhibitor overlap is not strong enough and the activator distributions of the two objects merge. This result differs from other simulations obtained in two-component systems with global coupling. In these systems the integral feedback forces the reproduction of activator as soon as the spots vanish and new pulses emerge from remaining perturbations. This is not possible for this set of parameters since D_w is rather low and thus w cannot simply be interpreted as global coupling. Using the same parameters but different initial conditions, annihilation of only one pulse has been observed, too. Though experimental scattering data are not yet available, the extinction of one of the spots is a typical experimental observation, when two spots come close to each other.

The third component offers a large range of possibilities. An example for a many-particle structure resulting from a pulse generation mechanism is shown in Fig. 4. For this calculation parameter κ_3 was increased to 10.5. Thus it is very easy to excite the system, and in fact new pulses are ignited in the refractory tail of an existing pulse. This process continues until the system is filled with moving spots. After some time, and due to mutual repulsion, these spots form a uniformly moving regular pattern, which is very similar to experimentally observed structures [1]. Another interesting direction is to examine moving spots for a higher diffusion-coefficient D_w of the second inhibitor. During the time of pulse interaction the third component acts like a global coupling since the spots are close together. Thus interesting features of the two-component case reappear in the progression of the scattering process. During the collision the spots merge

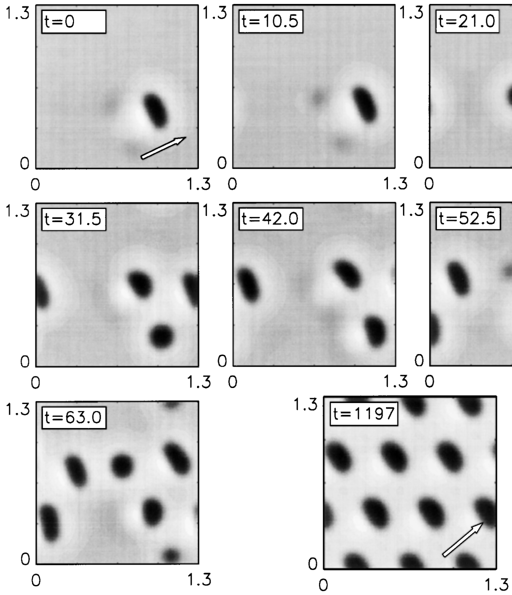

FIG. 4. Generation of new pulses in the refractory tail of the predecessor after increasing κ_3 to 10.5. After some time the system is filled with moving spots, which, after some transient oscillations, form a uniformly moving pattern. Parameters are like in Fig. 2 except that $L = 1.3$.

and divide up again. Apparently, by introducing a third component to a two-dimensional system, there is a great variety of new structures to be investigated.

Concluding, we return to the gas discharge experiment [1]. In this setup the charge-carrier density in the discharge gap can be considered as the activator since under the influence of the external field it may grow autocatalytically. The additional field, which is built up by these charges in the course of separation, is of opposite orientation. Hence it acts as an inhibitor, which is laterally spread due to the dielectric glas plates. Since electrons and ions have a different mobility, there are at least two time scales involved in the generation of the counterfield, justifying the use of two inhibitors. Of course, the real dynamics of the discharge is much more complex but we believe that our model offers at least a useful first approach. After submitting our manuscript we became aware of related work recently published by Zaikin [23]. In a relatively special three-component model, he numerically finds traveling spots and describes their behavior though for a limited time of evolution and without identifying a mechanism stabilizing the spot when moving. On the basis of his observations of spot-spot interactions and motivated by the relatively coarse simulation grid he applied, he gives an interpretation in terms of information processing in living systems of cellular structure. Whereas we consider this a rather speculative view, we strongly support his suggestion to design suitable experiments for the study of distinct traveling spots in two-dimensional systems and the numerous pattern formation phenomena implied. The set of RD equations we propose in this article is both simple and can directly be transformed to an equivalent circuit diagram similar to the one used in Refs. [3,11,13], which is closely related, e.g., to possible semi-conductor and dc gas discharge experiments. Thus, this field may now be entered successfully.

We thank E. Ammelt and I. Müller for stimulating discussions on their gas discharge experiments, the Höchstleistungsrechenzentrum (HLRZ) for providing CPU time on the paragon XP/S 10 computer at KFA, Jülich, and the Deutsche Forschungsgemeinschaft for financial support.

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