Fragmentation by Crack Branching

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(Received 4 November 1996)

Two-dimensional lattice models of crack branching give rise to fragmentation if disorder is introduced in the model. The resulting fragment-size distribution is analyzed within a simple analytical model and by numerical simulations. The analytical model gives, under rather general conditions, a power-law distribution over the entire size range. In the specific case studied, the exponent ranges from $-\infty$ to -0.5, depending on the stopping probability of cracks. The analytical results are consistent with the numerical simulations. [S0031-9007(97)03200-6]

PACS numbers: 46.30.Nz

Results of fragmentation processes are visible almost everywhere. Rock fragments, for example, are so common that the fragments have different names depending on their size: boulders, stones, gravel, sand, and dust, in decreasing order of size. The rather slow process of rock weathering is, of course, not the only fragmentation process in nature. Almost all kinds of explosions and collisions break pieces of matter into fragments. A common feature of almost all these fragmentation processes is that the fragment-size distribution is given by a power law in the small-size limit [1]. The origin of this power law has attracted much attention lately [2–5]. It cannot originate from randomly located microcracks that act as nucleation centers for cracks, as this mechanism would produce an exponential distribution of fragment sizes [1,6,7]. The exponential distribution is, of course, asymptotically a power law in the small-size limit, but power-law distributions found in experiments and numerical simulations usually extend over at least 1 order of magnitude [2–4,7].

It was argued in Ref. [7] that in two dimensions the exponential distribution, created by the merging of randomly nucleated cracks, holds in the large-fragment limit, while another mechanism, fragmentation by crack branching, dominates in the small-fragment limit. A fragment-size distribution which is a power law in the small-size limit and a more rapidly decreasing function in the large-size limit is also commonly found in both experiments and numerical simulations [2,4,7].

To test our previously reported suggestion [7] that a branching of cracks [8–12] is responsible for the power-law distribution of fragment sizes in the small-size limit, we employ here a type of lattice model recently used to investigate crack branching in a material without disorder [13–15]. To create fragments, however, disorder must be introduced in this lattice model. This is achieved by having randomly varying masses at the lattice sites.

The resulting lattice can be seen as a model of granular materials in which each lattice site corresponds to a grain, and lattice bonds correspond to elastic interactions between the grains. We wish to stress that our lattice model should be seen as a small piece of a larger object that is being fragmented. This justifies our assumption that the applied stress is constant both in space and in time. The dynamic stress field from, e.g., a rapid impact should be continuous if the sample contains no preexisting microcracks. The stress field is therefore constant in space if considered over a small enough area. As cracks propagate rapidly (and more or less perpendicularly to the local stress), the local stress field can be considered to be constant over the time it takes for a crack to propagate across such a small area.

The lattice models which we consider here are a square and a triangular lattice in which the lattice bonds are elastic beams with a square cross section w^2, length l, and Young's modulus E. Masses m_i are placed at the lattice sites i and the beams are massless. The masses m_i are chosen independently of each other from a uniform distribution $m(1 - d\delta_r)$, where d is the "strength" of the disorder, and δ_r is a stochastic variable whose values belong to the uniform distribution $[-0.5, 0.5]$. If $d < 2$, the disorder is "weak" in the sense of Hansen *et al.* [16]. The lattice is strained by an amount ϵ in the y direction, which in the square-lattice case is one of the principal bond directions, and in the triangular lattice perpendicular to one of the principal bond directions. The sites at the top and bottom edges of the lattices are constrained to remain at their original positions. In the square-lattice case the sites at the left and right edges are free to move without constraints. In the triangular lattice the sites at the left and right edges are only allowed to move in the y direction to avoid a global Poisson contraction of the lattice.

The dynamics of the lattices is calculated using a discrete form of Newton's equations of motion, including a small linear viscous dissipation term. The time evolution is calculated [17] by iteration of time steps starting from equilibrium at time $t = 0$. At $t = 0$, a few bonds in the middle of the left edge are suddenly removed. The other bonds break if the strain on them exceeds a threshold value, which in our case is taken to be a constant $\epsilon + \delta$ for the total displacement difference of the sites at the ends of the bond.

The fracture processes occurring in the two lattice models are illustrated in Figs. 1 and 2. Figures 1(b) and 2(b) display branching at small values of d, while Figs. 1(a) and 2(a) show the kind of fragments that are created for large values of d. It is evident, in Fig. 1(a), in particular, that crack branches merge to form fragments. From each merger of two branches usually only one branch continues to propagate. It is also possible that a branch stops spontaneously as can be seen in Figs. 1(a) and 2(a).

To get a more quantitative picture of this process we study a simple model of merging branches [18]. The model is sketched in Fig. 3. Branches appear on both sides of the central crack that propagates through the lattice perpendicularly to the external stress. These branches merge pairwise to form a single merged branch which continues to propagate. At each merger of two branches a fragment is formed. The merged branches continue to merge pairwise and thus form new fragments. The sizes of the fragments formed in this way, as sketched in Fig. 3, are given by

$$S = \sum_{n=0}^{B} 2^{2n}, \tag{1}$$

where $B = 0$ for the fragments formed in the first merger of branches, $B = 1$ in the second merger, and so on. If 2^A is the number of fragments formed in the first merger, then at each value of B, 2^{A-B} fragments are formed if there is no spontaneous stopping of cracks. If, due to crack stopping, only a fraction p of the possible fragments are formed at each value of B, then the number $N(B)$ of fragments formed at level B is

$$N(B) = 2^{A-B} p^B. \tag{2}$$

Notice that p is not necessarily a constant, but may depend on B. As calculated from Eqs. (1) and (2), the fragment-size distribution $N(S)$ of fragments of size S is given by

$$\ln[N(S)] = \left\{ -\frac{1}{2} + \frac{\ln[p(S)]}{\ln 4} \right\} \ln S + \ln\left[\frac{p(S)}{2} \right] \left(\frac{\ln 3}{\ln 4} - 1 \right) + A \ln 2 . \tag{3}$$

If p is constant, then Eq. (3) is of a power-law form, $N(S) \propto S^{-\alpha}$, where $\alpha = -1/2 + \ln p / \ln 4$.

As the number of fragments decreases with increasing B, a fragmentation process has a finite penetration depth if the number of fragments formed at $B = 0$ is limited. We define the penetration depth Δx as the distance from the central crack to the furthest edge of the largest fragment. Then, for a constant p, we have

$$\Delta x = 2^{\mu} - 1, \qquad \mu = \frac{\ln(2N_0) - \ln p}{\ln 2 - \ln p}, \tag{4}$$

where $N_0 = 2^A$ is the number of fragments formed in the first merger. Single cracks can propagate further but no more fragments are formed.

The model leading to Eq. (3) is an example of a fragmentation process that can be divided into steps (B) for which the size of the fragments formed is given by $S \propto a^B$, and the number of such fragments is $N \propto b^B$. Here a and b are model-dependent parameters. If there exists a range of B for which a and b are constant, the fragmentation process will create a fragment-size distribution of a power-law form; $N(S) \propto S^{\ln b / \ln a}$. These

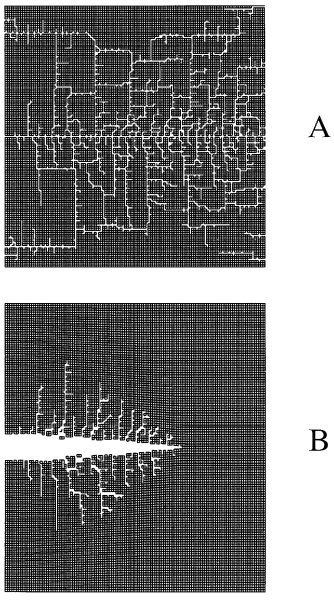

FIG. 1. (a) Crack pattern formed in a square lattice of size 120×120 after 700 time steps. Here $w = l = E = 1$, $m = 0.1$, and $d = 0.8$. (b) Branch pattern formed in a similar lattice, but with $d = 0.1$. The real displacements after 300 time steps are shown.

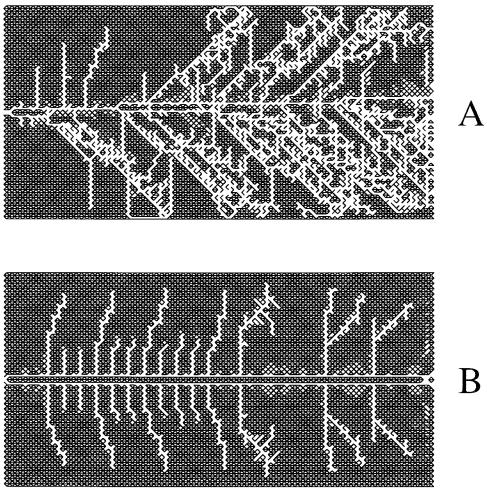

FIG. 2. (a) Crack pattern formed in a triangular lattice of size 160×80 after 400 time steps. Here $w = 0.8$, $E = 1$, $m = 0.1$, and $d = 0.3$. (b) Branch pattern formed in a similar lattice, but with $d = 0.05$.

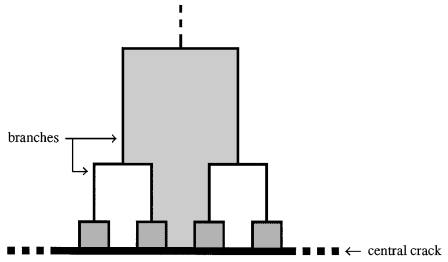

FIG. 3. Schematic picture of the fragmentation model. Fragments formed at stages $B = 0, 2$ are shaded.

conditions are not the most general ones, but they should nevertheless encompass a large class of fragmentation processes.

The total amount of elastic energy that is lost when the branches are formed is given by

$$dW \propto \int_0^B N(B')\sqrt{S(B')}\, dB'.$$

If the input of elastic energy is limited, then dW must also be finite for all B. This means that $b\sqrt{a} < 1$, which, when applied to Eq. (3), gives $p < 1$. In other words, all cracks will sooner or later stop spontaneously. A possible picture of the fragmentation process would then be that it proceeds with a more or less constant p until the energy is dissipated and p drops to zero. This picture would lead to a fragment-size distribution that has a power-law form at small sizes, and a distinct cutoff at a finite fragment size. If, on the other hand, there is an unlimited input of energy, no cutoff will occur.

A third possibility would be that p varies so that there is a constant probability that a crack will stop at each time a lattice bond is broken at the crack tip. Then p will decrease with increasing B, as longer cracks have to be formed for larger fragments. If $1 - \nu$ is the probability that a crack will stop at any lattice bond at the tip, then $p(B) = \exp[(2^{B+1} - 1)\ln\nu]$, and this means that

$$N(S) \propto (1/\nu)S^{-1/2}e^{\sqrt{3/4}\,\ln\nu\sqrt{S}}. \qquad (5)$$

This distribution becomes a power law for very small S (i.e., $S \approx 1$), but is essentially an exponential function of $\sqrt{S}$, and decreases therefore with increasing S much more rapidly than a power law.

We have used numerical simulations with the lattice models described above to test whether the fragment-size distribution is of the power-law form of Eq. (3) or whether it decreases faster [Eq. (5)]. In many experiments and numerical simulations the distributions obtained display power laws in the small-size limit, but decrease faster in the large-size limit [1,2,4]. If Eq. (3) is correct, then a more rapid decrease should be due to some mechanism other than the crack branching [7]. If, on the other hand, Eq. (5) is correct, then crack branching may explain the entire fragment distribution found in experiments and numerical simulations.

To get smooth distributions from the simulation data, we replace the distribution $N(S)$ by

$$M(S) \propto S^{-1}\int_S^\infty N(S')\, dS'.$$

The simulations give, as is evident from Fig. 4, for the square lattice a distribution that is very close to a power law for the entire size range. The simulated distribution is, on a log-log scale, a little curved, however, which suggests that p decreases slightly with increasing B, but the change in p is very small. For the triangular lattice, however, the distribution follows perfectly the power-law form for the entire size range.

Even though the model leading to Eq. (3) is very schematic, it seems to catch much of the qualitative behavior of the fragmentation process. In Fig. 1(a) the fragments are almost equally distributed along the central crack (except close to the left edge where the crack starts from). Most of the smallest fragments are close to the central crack and the fragments become, on the average, increasingly larger away from the central crack. If we fit the power α to the simulation results, we obtain $\alpha = -1.5$ for the square lattice, and $\alpha = -1.8$ for the

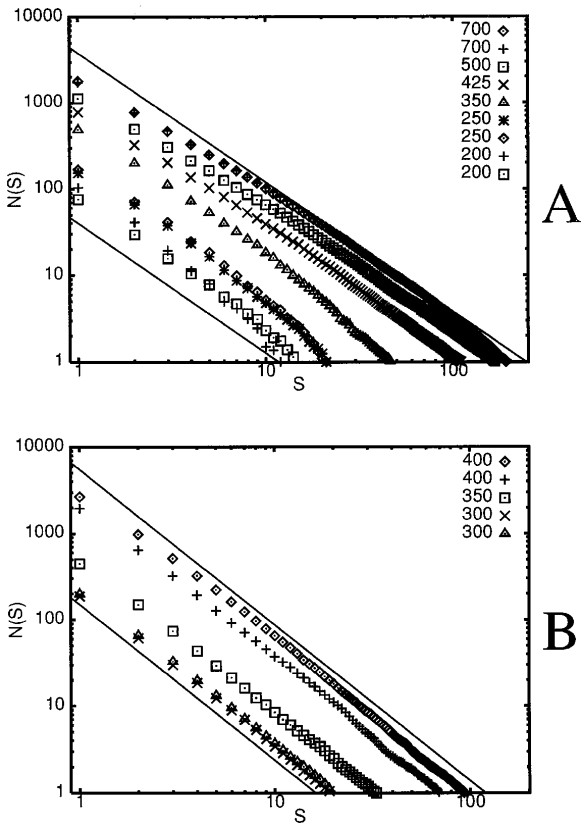

FIG. 4. Simulated fragment-size distributions for (a) square lattices, and (b) triangular lattices, for different numbers of time steps (200–700), and averaged over different values of d (0.1–1.5), and different values of δ (0.005–0.03). The symbols refer to the number of time steps used. The lines are $S^{-1.5}$ and $S^{-1.8}$ for (a) and (b), respectively.

triangular lattice. Both these powers are smaller than -0.5, which is consistent with Eq. (3).

We can also calculate the probability parameter p from the fitted values of α. Equation (3) gives $p = 0.25$ for the square lattice, and $p = 0.16$ for the triangular lattice. The parameter $1 - p$ is the probabilty that a crack stops spontaneously, which means that p is also the ratio of fragments created to the number of cracks that form dead ends. This quantity can be determined, e.g., from Fig. 1(a), and the result is $p \approx 0.275$. This is close to the value 0.25. For the triangular lattice [Fig. 2(a)], it is rather difficult to determine the exact number of dead ends, but there are clearly more of them than for the square lattice, which is consistent with the value $\alpha = 0.16$.

In summary, we have demonstrated that fragmentation of a lattice where the boundary conditions are such that only one central crack is formed, produces a fragment-size distribution through crack branching that is a power law for the entire size range. This suggests that the commonly found size distributions which are power laws in the small-size limit and exponentially decreasing functions in the large-size limit, originate from two different mechanisms. The small fragments are formed by the branching of propagating cracks, while the large fragments are formed by the merging of the randomly nucleated cracks [6]. The essential features of fragmentation by branching can be understood within a simple model of pairwise merging branches.

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