Two-loop QCD Corrections to Semileptonic b Decays at Maximal Recoil

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We present a complete $\mathcal{O}(\alpha_s^2)$ correction to the differential width of the inclusive semileptonic decay $b \to c l \nu_l$ at the kinematical point of vanishing invariant mass of the leptons, $q^2 = 0$. Together with the recently computed $\mathcal{O}(\alpha_s^2)$ correction at the upper boundary of the lepton invariant mass spectrum, this new information permits an estimate of the $\mathcal{O}(\alpha_s^2)$ effect in the total inclusive semileptonic decay width $b \to c l \nu_l$. We argue that the non–Brodsky-Lepage-Mackenzie (BLM) part of the $\mathcal{O}(\alpha_s^2)$ correction gives at most 1% correction to the inclusive semileptonic decay width $b \to c l \nu_l$. This significantly improves the credibility of extracting $|V_{cb}|$ from the inclusive semileptonic decays of the b hadrons. [S0031-9007(97)03164-5]

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Semileptonic decays of the b quarks provide the best opportunity to determine $|V_{cb}|$, a parameter of the Cabibbo-Kobayashi-Maskawa (CKM) matrix and a fundamental input parameter of the standard model. The current experimental limit [1]

$$|V_{cb}| = 0.036 \text{ to } 0.046 \quad (90\% \text{ C.L.}) \tag{1}$$

is based on measurements of the beauty hadron decays produced at the $\Upsilon(4S)$ resonance (by ARGUS and by CLEO II) and in Z-boson decays (by the four experiments at LEP). In the future large samples of the b hadrons collected at B factories (at SLAC and KEK) and at the hadron colliders will increase the statistical accuracy to a few percent level. To fully exploit the anticipated experimental improvement, the theoretical description of the b decay must be known with comparable precision.

There are two methods of extracting the value of $|V_{cb}|$, based on measurements of the exclusive decay $B \to \bar{D}^\star \bar{l} \nu_l$ and of the inclusive semileptonic decay width of b hadrons $\Gamma_{\rm sl}$. These two methods rely on very different theoretical considerations and experimental procedures and complement each other. Their merits and theoretical uncertainties are summarized, e.g., in Refs. [2–4]. One of the major sources of the theoretical error is the perturbative QCD corrections at the two-loop level. For the exclusive decays at the zero recoil point these corrections have recently been calculated [5]. This has significantly improved the accuracy of the theoretical prediction for the exclusive method.

In the case of the inclusive semileptonic decay width of the b hadrons $\Gamma_{\rm sl}$, the only known effects beyond one loop are those associated with the running of the strong coupling constant [6–8]. They are obtained by computing massless quark effects (Fig. 1) and then replacing the number of light flavors N_L by the combination in which it enters the one-loop β function $N_L - 33/2$. These so-called Brodsky-Lepage-Mackenzie (BLM) corrections [9] are expected to dominate the two-loop result; however, only a full calculation of the remaining diagrams will put this statement on a firm foundation.

Technically, the correction to the semileptonic decay width $\Gamma_{\rm sl}$ is obtained by fixing the invariant mass of the leptons q^2 and computing the differential width $d\Gamma_{\rm sl}/dq^2$ with desired accuracy. Integrating over q^2 within kinematical boundaries, one gets the inclusive semileptonic decay width of $b \to c l \nu_l$:

$$\Gamma_{\rm sl} = \int_0^{(m_b - m_c)^2} dq^2 \frac{d\Gamma_{\rm sl}}{dq^2} . \tag{2}$$

Going beyond the BLM approximation and computing complete $\mathcal{O}(\alpha_s^2)$ corrections remains a daunting task at present. In comparison with the zero recoil calculation the main difficulties are an additional kinematical variable describing the invariant mass of the leptons (q^2) and the presence of the real radiation of one and two gluons.

To circumvent these difficulties, we propose to estimate the deviations from the BLM predictions by performing complete $\mathcal{O}(\alpha_s^2)$ calcualtions for $d\Gamma_{\rm sl}/dq^2$ at two boundaries of integration in Eq. (2).

In fact, one of these calculations has already been done in Ref. [5] where $\mathcal{O}(\alpha_s^2)$ corrections to the transition $b \to c l \nu_l$ were calculated at the zero recoil limit. Since in this limit the radiation of real gluons is absent, the results of [5] provide $\mathcal{O}(\alpha_s^2)$ correction to $d\Gamma_{\rm sl}/dq^2$ at $q^2_{\rm max} = (m_b - m_c)^2$.

The purpose of this Letter is to present a calculation of the $\mathcal{O}(\alpha_s^2)$ corrections at $q^2_{\rm min} = 0$ which is the other boundary for the invariant mass of the leptons. With both

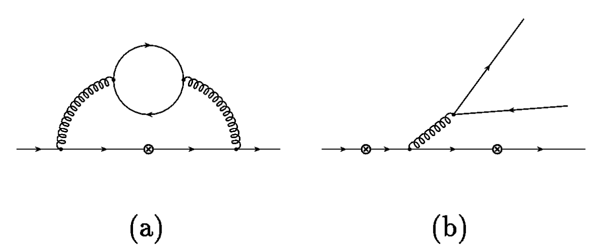

FIG. 1. Diagrams involving a light quark loop (a) or real pair emission (b). Symbols ⊗ mark places where the virtual W boson can possibly couple to the quark line.

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boundary points known we can estimate the deviation of the $\mathcal{O}(\alpha_s^2)$ corrections to the total inclusive semileptonic decay width of the b quark $\Gamma_{\rm sl}$ from the BLM prediction.

Taking the $q^2 = 0$ limit is important for the feasibility of this calculation. In this case the calculation of real radiation of one and two gluons is considerably simplified.

The reason why the real radiation at order $\mathcal{O}(\alpha_s^2)$ is difficult to calculate is that the particle in the initial state (the decaying b quark) carries a color charge and therefore can radiate. It is the presence of the massive propagators of this particle which makes the integrations over the phase space very tough. For this reason even the QED corrections to such well studied processes as the muon decay remain unknown at the two-loop level. The kinematical configuration in which $q^2 = 0$ and the quark in the final state is massive is the first case where the complete evaluation of the real radiation in the decay of a fermion turns out possible. Below we sketch the basic ideas of our approach; the technical details will be presented elsewhere.

The idea which permitted us to calculate the contribution due to the real radiation of one and two gluons is (qualitatively speaking) the expansion in the velocity of the final quark. Indeed, in the limit $m_c \to m_b$ the charm quark in the final state is a slowly moving particle, with spatial components of its momentum of the order of $m_b - m_c$, much smaller than its mass. The four momenta of gluons and of leptons (for $q^2 = 0$) are also of the order of $m_b - m_c$. It turns out that by a proper choice of the phase space variables one can systematically expand the amplitudes and the phase space in terms of $\delta \equiv (m_b - m_c)/m_b \ll 1$.

Some examples of the diagrams which contribute to the QCD corrections to the semileptonic decay of the b quark are shown in Fig. 2. Not shown are several other virtual corrections as well as diagrams obtained by permuting the gluon couplings to the quark line or by crossing the external gluon lines. In total there are about 80 Feynman diagrams which have to be evaluated.

We do not include the diagrams with three c quarks in the final state in our analysis. Since $3m_c$ is only marginally smaller than m_b, the contribution of such diagrams is strongly suppressed.

We parametrize the expansion using the variable $\delta = 1 - m_c/m_b$. In the first two nonvanishing orders (δ^3 and δ^4) only virtual corrections contribute [e.g., Figs. 2(a) and 2(b)]. The following two terms receive in addition contributions from diagrams with one loop and one real gluon emission [as in Figs. 2(c) and 2(d)], as well as from diagrams with two gluons resulting from a decay of a virtual gluon [Fig. 2(f)]. Only in the order δ^7 the contributions of a double gluon emission from the quark line show up [Fig. 2(e)]. This hierarchy can be traced back to the fact (evident in physical gauges) that the interaction of the slowly moving quarks with real gluons is proportional to the three velocity of the former.

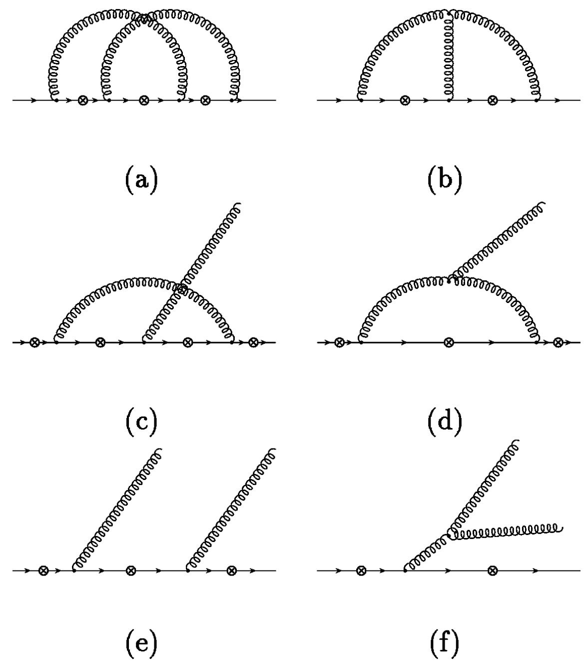

FIG. 2. Examples of the two-loop gluonic QCD corrections to the decay $b \to c l \nu_l$; (a),(b) virtual corrections; (c),(d) single gluon emission; (e),(f) emission of two gluons. Symbols $\otimes$ mark places where the virtual W boson can possibly couple to the quark line. The left hand side diagrams are QED-like, while the right hand side ones are purely non-Abelian.

In the case of two-loop virtual corrections as well as in the emission of two real gluons the expansion in δ means a Taylor expansion in the small external momenta of the leptons and gluons. Such an expansion does not lead to any spurious ultraviolet or infrared divergences. The situation is different in the case of the single gluon radiation in diagrams where there is in addition one virtual loop [Figs. 2(c) and 2(d)]. There a naive Taylor expansion in the external gluon momentum leads to artificial infrared divergences which correspond to the on-shell logarithmic singularities of the one-loop diagrams. Therefore a more sophisticated approach is needed and the recently developed method of "eikonal expansions" [10,11] is used.

To present our result we write the differential semileptonic decay width of the decay $b \to c l \nu$ at $q^2 = 0$ as

$$\left[\frac{d\Gamma_{\rm sl}}{dq^2}\right]_{q^2=0} = \Gamma_0\left[\Delta_{\rm Born} + \frac{\alpha_s}{\pi} C_F \Delta_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_F \Delta_2\right], \tag{3}$$

where $\Gamma_0 = \frac{G_F^2 m_b^3}{96\pi^3}|V_{cb}|^2$ and $\Delta_{\rm Born,1,2}$ describe the m_c/m_b dependence in various orders in the strong coupling constant.

Both $\Delta_{\rm Born} = (1 - m_c^2/m_b^2)^3$ and Δ_1 are known in a closed analytical form [12,13]. Δ_2 is the main result of the present Letter. For the purpose of presentation we divide it up into four contributions according to the color factors:

$$\Delta_2 = \delta^3[(C_F - C_A/2)\Delta_F + C_A\Delta_A + T_R N_L \Delta_L + T_R\Delta_H]. \quad (4)$$

The last term, Δ_H, describes the contributions of the massive b and c quark loops. Top quark contribution is suppressed by a factor $\sim m_b^2/m_t^2$ and has been neglected.

For the SU(3) group the color factors are $C_A = 3$, $C_F = 4/3$, $T_R = 1/2$. $N_L = 3$ is the number of the quark flavors whose masses have been neglected (u, d, and s).

We have computed the expansion coefficients of $\Delta_{F,A,L,H}$ up to δ^8, which for the physical value of the charm and bottom masses gives an estimated accuracy of our numerical predictions better than 1% (for $\delta = 1 - m_c/m_b \approx 0.7$).

In the present Letter we list the analytical results only up to δ^4, while the numerical evaluation is done using the expansions up to δ^8. Using the pole mass of the b and c quarks and expressing the one-loop corrections in terms of $\alpha_{\overline{MS}}(m_b^2)$ we find

$$\begin{aligned}
\Delta_A = {} & -\frac{355}{36} + \frac{2}{3}\pi^2 + \delta\left(\frac{89}{8} - \pi^2\right) + \delta^2\left(-\frac{2\,422\,517}{32\,400} + \frac{1708}{45}\ln(2\delta) - \frac{44}{9}\ln^2(2\delta) + \frac{8}{9}c_1 + \frac{257}{90}\pi^2\right) \\
& + \delta^3\left(\frac{2\,956\,607}{64\,800} - \frac{854}{45}\ln(2\delta) + \frac{22}{9}\ln^2(2\delta) - \frac{4}{9}c_1 - \frac{307}{180}\pi^2\right) \\
& + \delta^4\left(-\frac{5\,789\,957}{1\,323\,000} + \frac{4663}{4725}\ln(2\delta) + \frac{2}{5}\ln^2(2\delta) + \frac{4}{45}c_1 + \frac{412}{1575}\pi^2\right), \\
\Delta_F = {} & -\frac{23}{6} + \frac{8}{3}c_2 + \frac{8}{3}\pi^2 + \delta\left(\frac{23}{4} - 4c_2 - 4\pi^2\right) + \delta^2\left(\frac{1697}{360} - \frac{8}{3}\ln(2\delta) + \frac{22}{5}c_2 + \frac{359}{135}\pi^2\right) \\
& + \delta^3\left(-\frac{3347}{720} + \frac{4}{3}\ln(2\delta) - \frac{23}{15}c_2 - \frac{179}{270}\pi^2\right) \\
& + \delta^4\left(\frac{4\,957\,991}{396\,900} - \frac{1460}{189}\ln(2\delta) + \frac{16}{9}\ln^2(2\delta) + \frac{2}{7}c_2 - \frac{139}{600}\pi^2\right), \qquad (5) \\
\Delta_L = {} & \frac{14}{9} - \delta + \delta^2\left(\frac{82\,217}{4050} - \frac{544}{45}\ln(2\delta) + \frac{16}{9}\ln^2(2\delta) - \frac{16}{27}\pi^2\right) \\
& + \delta^3\left(-\frac{103\,667}{8100} + \frac{272}{45}\ln(2\delta) - \frac{8}{9}\ln^2(2\delta) + \frac{8}{27}\pi^2\right) \\
& + \delta^4\left(\frac{1\,322\,183}{496\,125} - \frac{2404}{1575}\ln(2\delta) + \frac{8}{45}\ln^2(2\delta) - \frac{8}{135}\pi^2\right), \\
\Delta_H = {} & \frac{460}{9} - \frac{16}{3}\pi^2 + \delta(-74 + 8\pi^2) + \delta^2\left(\frac{9821}{81} - \frac{344}{27}\pi^2\right) \\
& + \delta^3\left(-\frac{33\,883}{810} - \frac{32}{9}\ln(2\delta) + \frac{136}{27}\pi^2\right) + \delta^4\left(\frac{3754}{405} - \frac{154}{135}\pi^2\right),
\end{aligned}$$

with $c_1 = \frac{21}{2}\zeta_3 - \pi^2\ln(2\delta)$ and $c_2 = \frac{3}{2}\zeta_3 - \pi^2\ln 2$.

We now turn to the numerical analysis of our result. Here the issue of numerical values for the quark masses becomes important. It is safe to assume that the pole mass of the b quark lies between 4.6 and 5.1 GeV. The mass of the c quark is determined by $m_b - m_c$, obtained from the heavy quark effective theory (HQET) calculations [2–4,7]. We use $m_b - m_c \approx 3.45 \pm 0.10$ GeV where the error bar is rather conservative.

Accordingly, the numerical value of δ changes within the range of 0.65–0.77. The numerical values for the function Δ_2 become

$$\Delta_2 = -6.03,\ -7.45(4),\ -8.96\,, \quad (6)$$

for $\delta = 0.65$, 0.7, 0.75, respectively.

The error estimate, shown for the central value of $\delta = 0.7$, is obtained by multiplying the last computed term by 3, which corresponds roughly to summing up the remainder of the series in δ assuming constant coefficients. This procedure overestimates the error because the coefficients in fact decrease (there is at most a logarithmic divergence at $\delta = 1$ caused by neglected diagrams with three real c quarks in the final state).

Taken literally, the $\mathcal{O}(\alpha_s^2)$ corrections are quite large. However, as we will show below, the bulk of them is due to the BLM corrections.

The BLM prediction with four light flavors of quarks gives the following results:

$$\begin{aligned}
\Delta_2^{\mathrm{BLM}} &= -\delta^3\Delta_L T_R\left(\frac{33}{2} - 4\right) \\
&= -6.54,\ -8.15(6),\ -9.87, \qquad (7)
\end{aligned}$$

for $\delta = 0.65$, 0.7, 0.75, respectively.

By comparing the numbers in Eq. (7) with those in Eq. (6) we conclude that the BLM correction accounts for most of the effect. We estimate the residual correction by subtracting the BLM piece from the exact correction. We get a residual correction $(0.51, 0.7, 0.91) C_F (\alpha_s/\pi)^2$, which, using $\alpha_s(m_b) = 0.23$, gives numerically $0.5, 0.7, 0.8\%$ correction relative to the Born rate for $\delta =$ 0.65, 0.7, 0.75.

Therefore, we arrive at the conclusion that at the lower boundary of the invariant masses of leptons $q^2_{\min} = 0$, the BLM piece of the $\mathcal{O}(\alpha_s^2)$ correction represents the complete result with an excellent accuracy. The remaining correction does not exceed the value of 1% even accounting for an uncertainty in input parameters.

Finally, we estimate the $\mathcal{O}(\alpha_s^2)$ radiative corrections to the total semileptonic decay width of the b quark. In the BLM approximation such corrections have been calculated in Refs. [6,7]. Therefore, we are only interested in the deviations from the BLM approximation.

Our estimate of the non-BLM corrections to the inclusive width is based on the expectation that the largest deviation from BLM should occur at the maximal recoil limit, i.e., at $q^2 = 0$. To clarify this point, we note that the results of Ref. [5] imply that at zero recoil limit $[q^2_{\max} = (m_b - m_c)^2]$ the deviation of the exact result from the BLM approximation is very small. On the other hand, the results of this Letter show that at $q^2 = 0$ the non-BLM part of the correction grows with the decrease of the c-quark mass, i.e., with the increase in the phase space available for real gluon radiation. If one fixes the value of the c-quark mass, but varies instead the invariant mass of leptons q^2, the strongest emission of real gluons will occur at the maximal recoil point of the spectrum, at $q^2 = 0$. It is for this reason that we expect the largest discrepancy between the BLM prediction and the full correction at the lower end of the q^2 distribution, for $q^2 = 0$.

Turning to the estimate itself, from Ref. [5] we know that at $q^2_{\max} = (m_b - m_c)^2$ (zero recoil limit) the non-BLM correction to the differential width relative to the Born value is of the order of -0.1%. On the other end of the lepton invariant mass distribution the result of this Letter implies a slightly larger, but also tiny deviation below 1%. We note that the change of sign of the non-BLM corrections cancels part of their impact on the total width. Taking the absolute value of the larger of the corrections at the boundaries as an upper bound we conclude *that the non*-BLM *piece of the* $\mathcal{O}(\alpha_s^2)$ *corrections to the total semileptonic decay width* $b \to c l \nu_l$ *should not exceed the value of* 1%.

The value of second order correction to the inclusive width depends on the adopted definition of the quark masses. Our result is presented in terms of the pole masses, which is a convenient choice for the corrections not associated with the running of the coupling constant. It was argued in [14] that such parametrization leads to small higher-order non-BLM corrections. Our result confirms this expectation.

It is fair to say at this point that our estimate of the non-BLM piece of the corrections to the total inclusive semileptonic decay width based on the two boundary values cannot be considered as a rigorous proof. Keeping in mind that the complete calculation of the two-loop QCD corrections to the total decay width remains a very difficult task, a calculation of these corrections at some intermediate point q^2_{int} for the differential inclusive semileptonic decay width of the b quark is highly desirable. If such a calculation confirms that the non-BLM piece of the correction remains within the range set by its value on two boundaries, our estimate for the correction to the total semileptonic decay width of the b quark will be on a very safe ground.

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